The Dark Side of Loop Control Theory

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Course Agenda

- Introduction to Control Systems
- Shaping the Error Signal
- How to Implement the PID Block?
- The PID at Work with a Buck Converter
- Considering the Output Impedance
- Classical Poles/Zeros Placement
- Shaping the Output Impedance
- Quality Factor and Phase Margin
- What is Delay Margin?
- Gain Margin is not Enough
What is the Purpose of this Seminar?

- There have been numerous seminars on control loop theory
- Seminars are usually highly theoretical – link to the market?
- Control theory is a vast territory: you don't need to know everything!
- This 3-hour seminar will shed light on some of the less covered topics:
  - PID compensators and classical poles/zeros compensation
  - Output impedance considerations in a switching regulator
  - Understanding delay and modulus margins
  - In a 3-hour course, we are just scratching the surface...!
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What is a Closed-Loop System?

- A closed-loop system forces a variable to follow a setpoint
- The setpoint is the input, the controlled variable is the output
- French term is "enslavement": the output is slave to the input
- A car steering wheel is a possible example:

\[ u(t) \text{ control} \rightarrow \begin{cases} \theta \\ The \ input \ u \end{cases} \rightarrow \begin{cases} \theta \\ The \ output \ y \end{cases} \]
Representing a Closed-Loop System

- A closed-loop system can be represented by blocks
- The output is monitored and compared to the input

Any deviation between the two gives birth to an error $\varepsilon$
- This error is amplified and drives a corrective action
A Servomechanism or a Regulator?

Airplane elevator control is a servomechanism:
- The pilot imposes a mechanical position via the yoke

Car cruise control is a regulator:
- The speed is set, the car keeps it constant despite wind, etc.
Processing the Error Signal

- The error signal is processed through the compensator $G$
- We want the following operating characteristics:
  - Speed
  - Precision
  - Robustness

Since 1930

PID

Since 1930

$y(t)$

$u(t)$

overshoot

precision

speed

12.0u 36.0u 60.0u 84.0u 108u
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Where do You Shape the Signal?

- The compensator is the place where you apply corrections

\[ \varepsilon \rightarrow \text{Error processing} \rightarrow v_c \]

The compensator: \( G \)

- The compensator is built with an error amplifier:

- Op amp
- TL431
- OTA
The PID Compensator

A PID welcomes a Proportional block

\[ \varepsilon(t) \rightarrow k_p \rightarrow v_c(t) = k_p \varepsilon(t) \]

- \( k_p \) is high:
  - ✓ reaction speed
  - ❖ risks of overshoot
- \( k_p \) is low:
  - ❖ sluggish response

\[ k_p = 2.48 \]
The PID Compensator

A PID includes an Integrating block

\[ \varepsilon(t) \rightarrow \frac{k_p}{\tau_i} \int \varepsilon(t) dt \rightarrow v_c(t) = \frac{k_p}{\tau_i} \int_0^t \varepsilon(\tau)d\tau \]

Integral action:

- ✓ no static error
- ❇ slow response
- ❇ decreases stability
- ❇ large overshoots
The PID Compensator

- A PID offers a Derivative block

\[ \epsilon(t) \rightarrow k_p \tau_d \frac{d}{dt} \rightarrow v_c(t) = k_p \tau_d \frac{d\epsilon(t)}{dt} \]

Derivative action:
- ✓ perturbation variation is known: anticipation
- ✓ stabilizes the response
- ○ no static effect
The PID Compensator

The Derivative block anticipates the signal evolution and speed

\[
\Delta \varepsilon(t) = \frac{\varepsilon(t + \tau_d) - \varepsilon(t)}{\tau_d}
\]

\[
\varepsilon(t + \tau_d) \approx \varepsilon(t) + \tau_d \frac{d\varepsilon(t)}{dt}
\]

if \( \tau_d \) small

later \quad now
Combining the Blocks

- You can formulate the PID transfer function in different ways:

  - Differentiation: \( v_c(t) = \frac{d\varepsilon(t)}{dt} \rightarrow V_c(s) = \varepsilon(s)s \)
  - Integration: \( v_c(t) = \int \varepsilon(t) dt \rightarrow V_c(s) = \frac{\varepsilon(s)}{s} \)

- The standard form:

  \[
  G(s) = k_p \left( 1 + \frac{1}{s\tau_i} + s\tau_d \right)
  \]

- The parallel form:

  \[
  G(s) = k_p + \frac{k_i}{s} + s\tau_d
  \]

The derivative term cannot be physically implemented:

\[
\lim_{s \to \infty} s\tau_d = \infty
\]

Need a pole

\[
s\tau_d \rightarrow \frac{s\tau_d}{1 + \frac{s\tau_d}{N}}
\]
Combining the Blocks

- The transfer function becomes a filtered PID:

$$G(s) = k_p \left( 1 + \frac{1}{s \tau_i} + \frac{s \tau_d}{1 + \frac{s \tau_d}{N}} \right)$$

- If we develop, we obtain a more familiar expression:

$$G(s) = \frac{1 + s \left( \frac{\tau_d}{N} + \tau_i \right) + s^2 \left( \frac{\tau_d \tau_i}{N} + \tau_d \tau_i \right)}{s \frac{\tau_i}{k_p} \left( 1 + \frac{\tau_d}{N} s \right)}$$

- A double zero
- An origin pole
- A high-frequency pole
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Practical Implementation

- Sum up the output of each individual block:

\[ \sum \left( P k_p + I \frac{k_p}{\tau_i} + D k_p \tau_d \frac{d}{dt} \right) \]

- This is the parallel form of the PID
Practical Implementation

- This is the filtered standard form of the PID
Bridging a PID to a Type 3

A type 3 is implemented around an op amp

\[ G(s) = \frac{(1+s/\omega_1)(1+s/\omega_2)}{s(1+s/\omega_{po})\left(1+s/\omega_{p1}\right)\left(1+s/\omega_{p2}\right)} \]

\[ G(s) = \frac{1+s\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) + s^2\left(\frac{1}{\omega_1\omega_2}\right)}{s\omega_{po}\left(1+s/\omega_{p1}\right)\left(1+s/\omega_{p2}\right)} \]
Bridging a PID to a Type 3

Identify the terms and write the equations:

\[
G(s) = \frac{1 + s \left( \frac{\tau_d}{N} + \tau_i \right) + s^2 \left( \frac{\tau_d \tau_i}{N_1} + \tau_d \tau_i \right)}{s \frac{\tau_i}{k_p} \left( 1 + \frac{\tau_d}{N} \cdot s \right)}
\]

\[
G(s) = \frac{1 + s \left( \frac{1}{\omega_{z_1}} + \frac{1}{\omega_{z_2}} \right) + s^2 \left( \frac{1}{\omega_{z_1} \omega_{z_2}} \right)}{s \frac{1}{\omega_{po}} + \frac{s}{\omega_{po}}} \left( 1 + \frac{s}{\omega_{p_1}} \right)
\]

Four unknowns, four equations:

\[
\frac{\tau_d}{N} + \tau_i = \frac{1}{\omega_{z_1}} + \frac{1}{\omega_{z_2}}
\]

\[
\frac{\tau_d \tau_i}{N} + \tau_d \tau_i = \frac{1}{\omega_{z_1} \omega_{z_2}}
\]

\[
\frac{\tau_i}{k_p} = \frac{1}{\omega_{po}}
\]

\[
\frac{\tau_d}{N} = \frac{1}{\omega_{p_1}}
\]
Bridging a PID to a Type 3

- From Type 3 to PID:

\[ \tau_d = \frac{(\omega_{p_1} - \omega_{z_1})(\omega_{p_1} - \omega_{z_2})}{\left(\omega_{p_1} \omega_{z_1} + \omega_{p_1} \omega_{z_2} - \omega_{z_1} \omega_{z_2}\right) \omega_{p_1}} \]

\[ N = \frac{\omega_{p_1}^2}{\left(\omega_{p_1} \omega_{z_1} + \omega_{p_1} \omega_{z_2} - \omega_{z_1} \omega_{z_2}\right) \omega_{p_1}} - 1 \]

\[ \tau_i = \frac{\omega_{z_1} + \omega_{z_2} - 1}{\omega_{z_1} \omega_{z_2}} \]

\[ k_p = \frac{\omega_{po}}{\omega_{z_1}} - \frac{\omega_{po}}{\omega_{p_1}} + \frac{\omega_{po}}{\omega_{z_2}} \]

- From PID to Type 3:

\[ f_{z_1} = \frac{\tau_d - \sqrt{-4N^2\tau_d \tau_i + N^2\tau_i^2 - 2N\tau_d \tau_i + \tau_d^2 + N\tau_i}}{2\tau_d \tau_i (1 + N) \frac{2\pi}{2\pi}} \]

\[ f_{po} = \frac{N}{2\pi \tau_d} \]

\[ f_{z_2} = \frac{\tau_d + \sqrt{-4N^2\tau_d \tau_i + N^2\tau_i^2 - 2N\tau_d \tau_i + \tau_d^2 + N\tau_i}}{2\tau_d \tau_i (1 + N) \frac{2\pi}{2\pi}} \]

\[ f_{po} = \frac{k_p}{2\pi \tau_i} \]
Testing the Conversion

- Assume a type 3 compensator calculated for:
  \[ G_{f_c} = 1 \] at a crossover of \( f_c = 2740 \text{ Hz} \)
  \[ f_{z_1} = 200 \text{ Hz} \quad f_{z_2} = 600 \text{ Hz} \quad f_{p_1} = 21400 \text{ Hz} \quad f_{p_2} = 21400 \text{ Hz} \]
- The "0-dB crossover" pole is placed at:

\[
f_{po} = \frac{\sqrt{1 + \left( \frac{f_c}{f_{p_1}} \right)^2} \sqrt{1 + \left( \frac{f_c}{f_{p_2}} \right)^2}}{\sqrt{1 + \left( \frac{f_{z_1}}{f_c} \right)^2} \sqrt{1 + \left( \frac{f_c}{f_{z_2}} \right)^2}} \quad G_{f_c} f_{z_1} = 43.4 \text{ Hz} = 272 \text{ rd/s}
\]

- \( \tau_d = 194u \quad \tau_i = 1.05m \quad N = 25.6 \quad k_p = 0.287 \)
What is the "0-dB Crossover" Pole?

- If \( s \) appears as an isolated term in \( N(s) \), it is an origin pole

\[
G(s) = \frac{1}{s \left( \frac{s + 1}{\omega_p} \right)} \quad s = 0 \text{ is the origin pole}
\]

- If \( s \) is affected by a coefficient it is the "0-dB crossover pole"

\[
G(s) = \frac{1 + s/z_i}{A \left( 1 + s/\omega_p \right)} = \frac{s}{\omega_p} \left( \frac{s + 1}{\omega_p} \right) = \frac{\omega_p}{\omega_p} \left( \frac{s + 1}{s/z_i} \right) = \frac{G_0}{\left( 1 + s/\omega_p \right)}
\]

Dimension of a gain

Crossover pole

0 dB

\( \omega_p \)
We wanted a magnitude of 1 at 2.7 kHz

2.7 kHz, 0 dB
Maximum phase boost
Testing with SPICE

- This is the filtered form implementation

Parameters:

\[ i = \frac{1 + f_c^2/f_p1^2}{1 + f_c^2/f_p2^2} \]
\[ j = \frac{1 + f_z1^2/f_c^2}{1 + f_c^2/f_z2^2} \]
\[ W_i = (i/j) \cdot G \cdot f_p \cdot i^2 \]

Integration:

\[ T_D = 1.929e-004 \]
\[ N = 2.594e+001 \]
\[ T_I = 1.054e-03 \]
\[ KPF = 2.871e-01 \]
Testing with SPICE

\[ \angle G(s) \]

2.7 kHz, 0 dB
Maximum phase boost
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Stabilizing a Buck with a PID

We will use a PID to stabilize a voltage-mode Buck converter.
Small-Signal Response of the Buck

- The transfer function shows a resonance at 1.2 kHz
Compensating the Buck – Method 1

- We will explore three different methods for compensation:
  1. Shape closed-loop gain to make it a 2nd-order system
  2. Place poles and zeros to crossover at 10 kHz
  3. Shape the output impedance only

- Method 1 – derive the open-loop gain first

\[ T_{OL}(s) = G(s)H(s) \]
Compensating the Buck – Method 1

- The loop gain expression is that of the PID and $H(s)$

$$T_{OL}(s) = \frac{1+s\left(\frac{\tau_d}{N}+\tau_i\right)+s^2\left(\frac{\tau_d}{N}\tau_i+\tau_d\tau_i\right)}{N(s)}H_0$$

$$1+s\frac{\tau_i}{k_p}\left(1+\frac{\tau_d}{N}s\right)$$

- To simplify the expression, we can neutralize $D(s)$ by $N(s)$

Place a double zero at the double pole position:

$$1+s\left(\frac{\tau_d}{N}+\tau_i\right)+s^2\left(\frac{\tau_d}{N}\tau_i+\tau_d\tau_i\right) = \left(\frac{s}{\omega_0}\right)^2 + \frac{s}{Q_0\omega_0} + 1$$
Compensating the Buck – Method 1

The loop gain expression is now well simplified:

\[ T_{OL}(s) = \frac{H_0 \left(1 + \frac{s}{\omega_{z_1}} \right)}{s \frac{\tau_i}{k_p} \left(1 + \frac{\tau_d}{N} s \right)} \]

For a unity feedback control system, the closed-loop gain is:

\[ T_{CL}(s) = \frac{H_0 \left(1 + \frac{s}{\omega_{z_1}} \right)}{1 + \frac{H_0 \left(1 + \frac{s}{\omega_{z_1}} \right)}{s \frac{\tau_i}{k_p} \left(1 + \frac{\tau_d}{N} s \right)}} = \frac{1 + \frac{s}{\omega_{z_1}}}{1 + s \left(\frac{1}{\omega_{z_1}} + \frac{\tau_i}{H_0 k_p} \right) + s^2 \left(\frac{\tau_d \tau_i}{N H_0 k_p} \right)} \]
Compensating the Buck – Method 1

- We want a damped second-order response:

\[ 1 + s \left( \frac{1}{\omega_z} + \frac{\tau_i}{H_0 k_p} \right) + s^2 \left( \frac{\tau_d \tau_i}{N H_0 k_p} \right) = 1 + \frac{s}{\omega_c Q_c} + \left( \frac{s}{\omega_c} \right)^2 \]

Closed-loop denominator Second-order canonical form

- Choose a crossover frequency and a quality factor:

\[ Q_c = 0.5 \quad \text{Non-ringing response} \quad \omega_c = 27.3 \text{ rd/s} \rightarrow 10 \text{ kHz} \]
Compensating the Buck – Method 1

Four unknowns, four equations:

\[
\frac{1}{\omega_{z_i}} + \frac{T_i}{H_0k_p} = \frac{1}{\omega_cQ_c} \quad \frac{T_dT_i}{NH_0k_p} = \frac{1}{\omega_c^2} \quad \frac{T_d}{N} + T_i = \frac{1}{\omega_0Q_0} \quad \frac{T_d}{N_1} + T_i = \frac{1}{\omega_0^2}
\]

\[
T_d = \frac{Q_0Q_c^2\omega_{z_i}^2\omega_0^2 + Q_c^2\omega_{z_i}\omega_c^2 + Q_c^2Q_c^2\omega_c^4 - Q_c\omega_{z_i}^2\omega_0\omega_c - 2Q_0Q_c\omega_{z_i}\omega_c^3 + Q_0\omega_{z_i}^2\omega_c^2}{\omega_0\omega_c(\omega_c^2 - \omega_{z_i})(\omega_c^2 - \omega_{z_i} + Q_0Q_c\omega_z \omega_0)} = 1.116 \text{ms}
\]

\[
N = -\frac{Q_0Q_c^2\omega_{z_i}^2\omega_0^2 + Q_c^2\omega_{z_i}\omega_c^2 + Q_c^2Q_c^2\omega_c^4 - Q_c\omega_{z_i}^2\omega_0\omega_c - 2Q_0Q_c\omega_{z_i}\omega_c^3 + Q_0\omega_{z_i}^2\omega_c^2}{\omega_0\omega_c(\omega_c^2 - \omega_{z_i})(\omega_c^2 - \omega_{z_i} + Q_0Q_c\omega_z \omega_0)} = 72.4
\]

\[
T_i = -\frac{Q_c\omega_{z_i}^2 - \omega_{z_i}\omega_c + Q_0Q_c\omega_{z_i}\omega_0}{Q\omega_{z_i}\omega_c + Q_0Q_c\omega_{z_i}\omega_0} = 14.6 \mu s \quad k_p = -\frac{Q_c\omega_{z_i}(\omega_c^2 - \omega_{z_i}\omega_c + Q_0Q_c\omega_{z_i}\omega_0)}{H_0Q_0\omega_0(\omega_{z_i} - Q_c\omega_c)^2} = 0.178
\]
Compensating the Buck – Method 1

We can now compute our transfer functions in Mathcad®

Compensator response $G(s)$

Open-loop gain response $T_{OL}(s)$

$\phi_m = 80^\circ$
Compensating the Buck – Method 1

- The closed-loop system is perfectly compensated

\[ T_{CL}(s) = \frac{T_{OL}(s)}{1 + T_{OL}(s)} \]

11.8 kHz
-3 dB
Compensating the Buck – Method 1

We can test the compensation with SPICE

parameters

\[ fc = 10k \]
\[ G = 5 \]
\[ V_{in} = 10 \]
\[ V_{peak} = 2 \]
\[ G = 10^{-G_{fc}/20} \]
\[ p = 3.14159 \]
\[ f_{z1} = 1.2k \]
\[ f_{z2} = 1.2k \]
\[ f_{p1} = 10k \]
\[ f_{p2} = 50k \]
\[ W_{z1} = 2\pi f_{z1} \]
\[ W_{z2} = 2\pi f_{z2} \]
\[ W_{p1} = 2\pi f_{p1} \]
\[ W_{p2} = 2\pi f_{p2} \]

\[ e = (W_{p1} - W_{z1})(W_{p1} - W_{z2}) \]
\[ f = W_{p1}W_{z1} + W_{p1}W_{z2} - W_{z1}W_{z2} \]
\[ T_{dd} = e/(fW_{p1}) \]
\[ T_{ii} = (W_{z1} + W_{z2})/(W_{z1}W_{z2}) - (1/W_{p1}) \]
\[ k_{pff} = (W_{p1}W_{z1} - W_{p1}W_{z2} - W_{p1}W_{z1})/2W_{p1} \]

\[ T_{dd} = 1.116m \]
\[ T_{ii} = 14.6u \]
\[ k_{pff} = -0.178 \]
\[ N = 72.4 \]
We can test the compensation with SPICE

Perfect compensation dude!
Compensating the Buck – Method 1

- We have a stable but oscillatory response!

\[ \zeta = \frac{1}{1 + \frac{2\pi}{\ln\left(\frac{20}{45}\right)}} \approx 0.128 \]

\[ Q = \frac{1}{2\zeta} \approx 3.9 \]

\[ \Delta I_{out} = 1 \text{ A in } 100 \mu\text{s} \]
Compensating the Buck – Method 1

- Bode or Nyquist do not predict the oscillatory response
Compensating the Buck – Method 1

- We compensated the $V_{out}$ to $V_{ref}$ path only!

$$V_{ref}(s) \rightarrow G_{PWM}(s) \rightarrow H(s) \rightarrow I_{in}(s)Z_{out}(s) \rightarrow V_{out}(s)$$

$$V_{ref}(t) \rightarrow TCL(s) \rightarrow V_{out}(t)$$
Compensating the Buck – Method 1

- The output response is as expected

- Where is the issue coming from then?
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Compensating the Buck – Method 1

- In reality, $V_{\text{ref}}$ is fixed: "we have a regulator, stupid!"

$$V_{\text{ref}}(s) \rightarrow \underbrace{G(s)}_{+} \rightarrow \underbrace{H(s)}_{=} \rightarrow V_{\text{out}}(s)$$

- Because the system is linear, superposition applies

$$V_{\text{out}}(s) = V_{\text{ref}}(s) \frac{T_{\text{OL}}(s)}{1+T_{\text{OL}}(s)}$$

$$V_{\text{out}}(s) = I_{\text{out}}(s) Z_{\text{out,OL}}(s) - V_{\text{out}}(s) T_{\text{OL}}(s)$$

$$V_{\text{out}}(s) = V_{\text{out1}}(s) + V_{\text{out2}}(s) = V_{\text{ref}}(s) \frac{T_{\text{OL}}(s)}{1+T_{\text{OL}}(s)} - I_{\text{out}}(s) \frac{Z_{\text{out,OL}}(s)}{1+T_{\text{OL}}(s)} Z_{\text{out,CL}}$$

- During the load step, $\dot{V}_{\text{ref}} = 0$: $Z_{\text{OUT}}$ fixes the response!
Compensating the Buck – Method 1

- What matters is the output impedance $Z_{out, CL}$.

- What is the output impedance of a buck converter?

It follows the form

$$Z_{out}(s) = R_0 \frac{N(s)}{D(s)}$$

No dimension
Compensating the Buck – Method 1

- The output impedance is a transfer function
  \[ Z_{out}(s) = \frac{V_{out}(s)}{I_{out}(s)} \]
  - response
  - excitation

- Let's find the term \( R_0 \) in dc: open caps, short inductors
  \[ R_0 = r_L \parallel R_{load} \]

- The zeros cancel the response

\[ L \]
\[ r_L \]
\[ R_{load} \]
\[ r_C \]
\[ C_{out} \]
\[ sL + r_L = 0 \quad r_L \left( \frac{L}{r_L} + 1 \right) = 0 \]
\[ \omega_z = \frac{r_L}{L} \]
\[ \frac{1}{sC_{out}} + r_C = 0 \quad 1 + sr_C C_{out} = 0 \]
\[ \omega_{z2} = \frac{1}{r_CC_{out}} \]
Compensating the Buck – Method 1

- The denominator is solely dependent on the structure
  - It is independent from the excitation: set it to zero!
- There are two storage elements: this is a 2nd-order network
  \[ D(s) = 1 + a_1 s + a_2 s^2 = 1 + \frac{s}{\omega_0 Q} + \left( \frac{s}{\omega_0} \right)^2 \]
- \( D \) must be dimensionless thus: \( a_1 \equiv (Hz)^{-1} \) \( a_2 \equiv (Hz)^{-2} \)
  - The two possible terms for \( a_1 \) are \( \tau_1 + \tau_2 \)
  - The two possible terms for \( a_2 \) are \( \frac{\tau_1 \tau_2}{\tau_1^' \tau_2^'} \)
Compensating the Buck – Method 1

- For \( a_1 \) look at the resistance \( R \) driving \( L \) and \( C \)
  
- Look at the driving impedance at \( L \) while \( C \) is in its dc state
- Look at the driving impedance at \( C \) while \( L \) is in its dc state

\[
R = r_L + R_{load} \quad R = (r_L \parallel R_{load}) + r_C
\]

\[
\text{... } s \left( \frac{L}{r_L + R_{load}} + C\left[ (r_L \parallel R_{load}) + r_C \right] \right)
\]

\[
\downarrow \tau_1 \quad \downarrow \tau_2
\]
Compensating the Buck – Method 1

- how $\tau_1$ (involving $L$) combines with $\tau'_2$ (involving $C$)?
- how $\tau_2$ (involving $C$) combines with $\tau'_1$ (involving $L$)?

- Look at the driving impedance at $C$ while $L$ is in its HF state
- Look at the driving impedance at $L$ while $C$ is in its HF state

If we chose $\tau_1 = \frac{L}{r_L + R_{load}}$,

$\tau'_2 = C(r_c + R_{load})$

Same result

$\tau'_1 = \frac{L}{r_L + (R_{load} \parallel r_c)}$

$\tau_1 \tau'_2 = \tau'_1 \tau_2$
Compensating the Buck – Method 1

- We have our denominator!

\[ D(s) = 1 + s \left( \frac{L}{r_L + R_{load}} + C \left( (r_L \parallel R_{load}) + r_C \right) \right) + s^2 \left( \frac{LC}{r_L + R_{load}} \right) \left( r_C + R_{load} \right) \]

- The complete transfer function is now:

\[ Z_{out}(s) = \left( r_L \parallel R_{load} \right) \frac{1 + s \frac{L}{r_L}}{1 + s \left( \frac{L}{r_L + R_{load}} + C \left( (r_L \parallel R_{load}) + r_C \right) \right) + s^2 \left( \frac{LC}{r_L + R_{load}} \right) \left( r_C + R_{load} \right)} \]

See "Fast Analytical Techniques" from Vatché Vorpérian, Cambridge Press
Compensating the Buck – Method 1

☐ It can be put under the following form:

\[ Z_{\text{out}}(s) = R_0 \left( \frac{1 + s/\omega_{z_1}}{1 + \frac{s}{\omega_0 Q} + \left( \frac{s}{\omega_0} \right)^2} \right) \left( 1 + s/\omega_{z_2} \right) \]

☐ Where we can identify the terms:

\[ R_0 = r_L \parallel R_{\text{load}} \quad \omega_{z_1} = \frac{r_L}{L} \quad \omega_{z_2} = \frac{1}{r_C C_{\text{out}}} \]

\[ \omega_0 = \frac{1}{\sqrt{L C_{\text{out}}}} \sqrt{\frac{r_L + R_{\text{load}}}{r_C + R_{\text{load}}}} \quad Q = \frac{L C_{\text{out}} \omega_0 \left( r_C + R_{\text{load}} \right)}{L + C_{\text{out}} \left( r_L r_C + r_L R_{\text{load}} + r_C R_{\text{load}} \right)} \]
Compensating the Buck – Method 1

- If we now plot the output impedance, we see peaking.
- For an non-oscillatory response, the peaking must be damped!

It's not, this is where the problem comes from.
Compensating the Buck – Method 1

- We organized a gain deficit right at the resonance!

- To tame the peaking, we must have gain at $f_0$

- How much do we peak at $f_0$?

\[
|Z_{\text{out,max}}(\omega_0)| = R_0 \cdot \frac{\sqrt{1 + \left(\frac{\omega_0}{\omega_1}\right)^2} \sqrt{1 + \left(\frac{\omega_0}{\omega_2}\right)^2}}{\sqrt{1 - \omega_0^2 \left(\frac{LC_{\text{out}}}{r_c + R_{\text{load}}}\right)^2} + \left[\frac{\omega_0}{r_c + R_{\text{load}}} + C_{\text{out}}\left(r_c + r_L + R_{\text{load}}\right)\right]^2}
\]
Compensating the Buck – Method 1

- We can impose a magnitude to stay below \( r_C \)
  - evaluate the needed gain to fulfill this goal:

\[
\frac{Z_{\text{out, max}} (f_0)}{1 + T_{OL} (f_0)} \leq r_C \quad \approx \quad \frac{Z_{\text{out, max}} (f_0)}{|T_{OL} (f_0)|} \leq r_C \quad \Rightarrow \quad |T_{OL} (f_0)| \geq \frac{Z_{\text{out, max}} (f_0)}{r_C}
\]

Closed-loop output impedance

- Applying the numerical values of the buck:

\[
|T_{OL} (f_0)| \geq \frac{1.12}{70m} \geq 16 \text{ or } 24 \text{ dB}
\]

- Is this enough to obtain a ringing-free response?
Compensating the Buck – Method 1

- No, ringing is reduced but not eliminated

- The peaking in the output impedance is still there!
- The notched zeros are the cause for the gain dip at $f_0$
  ➢ We must find a different compensation method
Another (Bad) Example

- Can we crossover at 10 Hz according to this plot?

- We have no phase lag at 10 Hz, a type 1 could do?

\[ |H(10\text{ Hz})| = 14\text{ dB} \]
\[ \text{arg } H(10\text{ Hz}) = 0^\circ \]
Rolling-off the BW at Low Frequencies

- SPICE gives us the open-loop gain snapshot

```
parameters
Vout=5V
Rupper=10k
Gfc=14
G=10^(-Gfc/20)
pi=3.14159
C1=1/(2*pi*fc*Rupper)
f0=1/(2*pi*C1*Rupper)
```

-6 dB
PWM gain
The Open-Loop Gain Looks Good…

- The type 1 confirms our 0-dB crossover frequency

![Graph showing open-loop gain characteristics](image)

- $\theta_m = 90^\circ$
- $f_c = 10\, \text{Hz}$
- $GM = 24\, \text{dB}$
As Expected: It is Ringing!

- The load step reveals a ringing ac output

\[ V_{\text{out}}(t) \]

\[ \Delta I_{\text{out}} = 2 \text{ A} \]

\[ V_{\text{cp}}(t) \]

\[ V_{\text{cp}}(t) \text{ is first order} \]
Good dc Coupling, Weak ac Coupling

- $H_1$ is stable per Bode analysis, but $H_2$ is out of the loop…

- Oscillations are NOT due to the loop!

- The dc is fed back via the loop but not the ac…

$V_{in} \rightarrow H_1(s) \rightarrow H_2(s) \rightarrow V_{out}(s)$

$G(s)$

Loose coupling in ac, no signal transmission $> f_c$…

$T(s)$

dc $f < f_c$

ac $f > f_c$

Again, an Undamped RLC Network…

- No gain at resonance: the \( RLC \) network runs open loop

- The system cannot reduce the \( Q \) at the resonant frequency

\[
\begin{align*}
\text{Gain action} \\
\text{output filter } f_0 \\
\text{Properly compensated, } f_c = 4 \text{ kHz}
\end{align*}
\]
Course Agenda

- Introduction to Control Systems
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Compensating the Buck – Method 2

- In this method, we will focus on two parameters:
  - crossover frequency $f_c$
  - phase margin $\phi_m$

- First, look at the ac response of the power stage:
Compensating the Buck – Method 2

- The peaking in $H$ brings a severe phase lag at $f_0$
  - stay away from $f_0$, pick $f_c$ at least 10 times above (10 kHz)
  - extract the phase/magnitude of $H$ at 10 kHz:

$$|H(10\text{kHz})| = -20\text{dB} \quad \angle H(10\text{kHz}) = -132^\circ$$

- The compensator $G$ must shape the loop gain $T_{OL}$ by
  - providing a high dc gain for precision: place an origin pole
  - reducing the phase lag at 10 kHz to provide a $\phi_m$ of 70°

---

What compensation type do we need?
Compensating the Buck – Method 2

- The origin pole brings a permanent phase lag of 90°
- added to the op amp inversion of -180°, we have -270°
- total phase (op amp and \( H \)) must be -360 + 70° = -290°
- the needed phase boost at \( f_c \) is thus:

\[
boost = \varphi_m - \angle H (f_c) - 90 = 70 + 132 - 90 = 112°
\]

![Graphs showing phase margins for different boost types](image)

Type 1 – no boost  
Type 2 – up to 90°  
Type 3 – up to 180°
Compensating the Buck – Method 2

- type 3: an origin pole, a double zero and 2 poles
  - this is our PID compensator!

\[
G(s) = -\frac{s}{\omega_p(1 + s/\omega_p)(1 + s/\omega_z)} = -\frac{1 + \frac{\omega_z}{s}}{\frac{\omega_p}{\omega_z}}
\]

- The magnitude is derived as:

\[
|G(f)| = \frac{f_{po}}{f_{z1}} \sqrt{1 + \left(\frac{f_{z1}}{f}\right)^2} \sqrt{1 + \left(\frac{f}{f_{z2}}\right)^2}
\]

- The argument is found to be:

\[
\arg G(f) = \arg N - \arg D
\]

\[
\arg N = \arctan \left(\frac{f_{z1}}{f}\right) - \pi + \arctan \left(\frac{f}{f_{z2}}\right)
\]

\[
\arg D = \arctan \left(\frac{f}{f_{p1}}\right) + \arctan \left(\frac{f}{f_{p2}}\right)
\]
Compensating the Buck – Method 2

- Place the double zero at $f_0$, the second pole at $F_{sw}/2$
- The 0-dB crossover pole is adjusted to provide +20 dB at $f_c$
- The first pole is adjusted to provide the right $\varphi_m$

$$\arg G(f_c) = \arctan \left( -\frac{f_{z_1}}{f_c} \right) - \pi + \arctan \left( \frac{f_c}{f_z} \right) - \arctan \left( \frac{f_c}{f_{p_1}} \right) - \arctan \left( \frac{f_c}{f_{p_2}} \right)$$

$$f_{p_1} = -\frac{f_c}{\tan \left( \arg G + \tan^{-1} \left( \frac{f_c}{f_{p_2}} \right) + \tan^{-1} \left( \frac{f_{z_1}}{f_c} \right) - \tan^{-1} \left( \frac{f_c}{f_z} \right) \right)} = 10.8 \text{ kHz}$$
Compensating the Buck – Method 2

- The 0-dB crossover pole is adjusted to give 20 dB at 10 kHz:

\[
f_{po} = \left| G(f_c) \right| f_{z_1} \sqrt{1 + \left( \frac{f_c}{f_{p_1}} \right)^2} \sqrt{1 + \left( \frac{f_c}{f_{p_2}} \right)^2} \approx 2\,\text{kHz}
\]

- The final configuration is as follows:

\[
\text{It should be ok to damp } Z_{out}
\]

\[
|T_{oc}(1.2\,\text{kHz})| = \left| G(1.2\,\text{kHz}) \right| \left| H(1.2\,\text{kHz}) \right| = 37\,\text{dB}
\]
Compensating the Buck – Method 2

- Enter the PID coefficients from the poles/zeros positions
  \[ \tau_d = 55.5u \quad \tau_i = 250u \quad N = 3.76 \quad k_p = 3.1 \]

- More than 30 dB at \( f_0 \) and absolutely no peaking in \( Z_{out} \)
Compensating the Buck – Method 2

The transient response with a 0.1-A load step is excellent

\[
\begin{align*}
V_{out}(t) & \\
\approx 8 \text{ mV} & \\
Z_{out,CL}(10k) & = -21 \text{ dBΩ} = 84 \text{ mΩ}
\end{align*}
\]
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Compensating the Buck – Method 3

- The capacitor stray elements affect the transient response

\[ V_{out}(t) \]

\[ I_{out}(t) \]

\[ \Delta I_{out} \]

\[ S = \frac{\Delta I_{out}}{\Delta t} \]

Compensating the Buck – Method 3

- During a step load, the converter fights the current change

\[ V_{\text{out}}(t) = \frac{V_{\text{max}} - V_{\text{min}}}{2} \]

- Traditional compensation:
  - inductive output impedance
  - resistive output impedance
  - Limited excursion

- Adaptive Voltage Positioning
  - Full-span excursion

\[ V_{\text{max}} \quad V_{\text{out}}(t) \quad V_{\text{min}} \]
Compensating the Buck – Method 3

- Whatever the gain, $Z_{out}$ meets the open-loop value beyond $f_c$

- Make the output impedance equal to $r_C$ along the freq. range
Compensating the Buck – Method 3

- How to force the output impedance to be resistive?

\[
Z_{out,CL} = \frac{Z_{out,OL}}{1 + T(s)} = R_0 \frac{1}{1 + H_0} \frac{1}{\left(1 + \frac{s}{\omega_{z_1}}\right) \left(1 + \frac{s}{\omega_{z_2}}\right) + \frac{s}{\omega_0} + 1} \frac{1}{\left(1 + \frac{s}{\omega_0 Q}\right) + 1} G(s)
\]

Extract \(G(s)\) to have:

\[
Z_{out,CL} = r_C
\]
Compensating the Buck – Method 3

- Once \( G(s) \) is extracted, what filter is that?

\[
G(s) = \frac{R_0 \left(1 + \frac{s}{\omega_{z_2}}\right)}{r_c G_0} - \frac{s}{\omega_0^2} + \frac{s}{Q \omega_0} + 1
\]

\[
G(s) = K_0 \left(1 + \frac{s}{\omega_{z_2}}\right) + \frac{s}{\omega_p G}
\]

\[
\omega_p = \omega_{z_2}
\]
Compensating the Buck – Method 3

- Some parameter identification is now needed:

\[ K_0 = \frac{r_L - r_C}{H_0 r_C} = 1.8 \quad a = \frac{r_L}{\omega_z^2} - \frac{r_C}{\omega_0^2} = 47.7 \quad b = r_L \left( \frac{1}{\omega_z} + \frac{1}{\omega_0} \right) - \frac{r_C}{Q \omega_0} = 74.2 \]

\[ c = r_L - r_C = 0.27 \quad f_{p_i} = f_z = 24 \text{ kHz} \quad f_{z_0} = \frac{b - \sqrt{b^2 - 4ac}}{4\pi a} = 580 \text{ Hz} \]

YAO et al., "Design Considerations for VRM Transient Response Based on the Output Impedance", IEEE Proceedings, 2003
Compensating the Buck – Method 3

- The following op amp architecture will do the job

\[ V_{out}(s) \]

\[ \frac{v_c(s)}{V_{out}(s)} = \frac{R_1}{Z_1} \]

\[ Z_1 = \frac{R_2 \left( R_3 + \frac{1}{sC_1} \right)}{R_2 + \left( R_3 + \frac{1}{sC_1} \right)} \]

\[ G(s) = -\frac{R_2}{R_1} \frac{1 + sC_1 \left( R_2 + R_3 \right)}{1 + sR_3C_1} \]

\[ K_0 = \frac{R_2}{R_1} \quad \omega_z = \frac{1}{C_1 \left( R_2 + R_3 \right)} \quad \omega_n = \frac{1}{R_3C_1} \]
Compensating the Buck – Method 3

A test fixture is assembled using a buck averaged model.

parameters

- \( V_{in} = 10 \)
- \( V_{peak} = 2 \)
- \( \pi = 3.14159 \)
- \( f_z = 580 \)
- \( f_p = 24k \)
- \( W_z = 2^p \pi f_z \)
- \( W_p = 2^p \pi f_p \)
- \( k_v = 1.8 \)
- \( R_1 = 10k \)
- \( R_2 = R_1 / k_v \)
- \( R_3 = R_1 W_z / (k_v (W_p - W_z)) \)
- \( C_1 = (k_v (W_p - W_z)) / (R_1 W_p W_z) \)

\[ R_{f} = R_{1} \]

\[ R_{8} = R_{0} \]

\[ R_{10} \]

\[ R_{12} \]

\[ R_{7} (R_{8} + 1.47) \]

\[ C_{2} \]

\[ C_{1} \]

\[ W_{z} = 2^p \pi f_{z} \]

\[ W_{p} = 2^p \pi f_{p} \]

\[ k_v = 1.8 \]

\[ R_{1} = 10k \]

\[ R_{2} = R_{1} / k_v \]

\[ R_{3} = R_{1} W_{z} / (k_v (W_{p} - W_{z})) \]

\[ C_{1} = (k_v (W_{p} - W_{z})) / (R_{1} W_{p} W_{z}) \]
Compensating the Buck – Method 3

- After stabilization we have good margins with a 20-kHz $f_c$

![Graph showing $|T(s)|$ and $\angle T(s)$](image)

- The output impedance is purely resistive!
- But the dc gain is low: line and load regulation problems!
Compensating the Buck – Method 3

- Also, the gain expression $K_0$ can be a problem:

$$K_0 = \frac{r_L - r_C}{G_0 r_C} > 0 \rightarrow r_L > r_C$$

- VM, fixed frequency, is not the best for $Z_{out}$ resistive shaping
Compensating the Buck – Method 3

- Going current-mode, fixed frequency is one way to go

- Use the PWM switch model in current-mode for $Z_{out}$
Compensating the Buck – Method 3

- The large-signal model combines two current-sources

\[ I_L(t) \]

\[ \frac{v_c}{R_i} \]

\[ v_{out}/L \]

\[ \Delta I_L \]

\[ I_c \]

\[ \frac{v_c}{R_i} - \frac{\Delta I_L}{2} \]

\[ I_2 = \frac{v_c}{R_i} - \frac{(1 - D)T_{sw}}{2L}V_{out} \]
Compensating the Buck – Method 3

- For ac study, we must obtain a small-signal model

\[
I_2(V_{out}, V_c) = \frac{V_c}{R_i} - \frac{(1-D)T_{sw}}{2L} V_{out} = \frac{V_c}{R_i} - \left( \frac{1-V_{out}}{V_{in}} \right) T_{sw} V_{out}
\]

- Calculate the partial derivative coefficients to \( v_{out} \) and \( v_c \)

\[
\frac{\partial I_2}{\partial V_c} \hat{v}_c + \frac{\partial I_2}{\partial V_{out}} \hat{v}_{out} = \frac{\hat{v}_c}{R_i} + \frac{T_{sw}}{2L} \left( \frac{2V_{out}}{V_{in}} - 1 \right) \hat{v}_{out} \rightarrow \text{gm}
\]

- Update the schematic with this linear source
  - as \( R, L \) and \( C \) are also linear, Laplace applies
Compensating the Buck – Method 3

- We look at the output impedance closed-loop

\[ v_{out}(s) = \frac{v_c(s)}{R_L} + gm \cdot v_{out}(s) \]

\[ i_{out}(s) = \frac{v_c(s)}{R_C} \]

\[ G(s) = \frac{v_{out}(s)}{v_c(s)} \]

\[ gm = \frac{T_{sw}}{2L} \left( \frac{2V_{out}}{V_{in}} - 1 \right) \]
Compensating the Buck – Method 3

- We can apply the superposition theorem:

\[
v_{out}(s) = \left(\frac{v_c(s)}{R_i} + gm \cdot v_{out}(s)\right)\left(R_{load} \parallel \left(r_C + \frac{1}{sC_{out}}\right)\right) \quad \text{if } i_{out} \text{ is 0}
\]

\[
v_{out}(s) = -i_{out}(s)\left(R_{load} \parallel \left(r_C + \frac{1}{sC_{out}}\right)\right) I_2 \text{ is 0}
\]

- Considering \( v_c(s) = -G(s)v_{out}(s) \) we have:

\[
v_{out}(s) = v_{out}(s)\left(gm - \frac{G(s)}{R_i}\right)\left(R_{load} \parallel \left(r_C + \frac{1}{sC_{out}}\right)\right) - i_{out}(s)\left(R_{load} \parallel \left(r_C + \frac{1}{sC_{out}}\right)\right)
\]

\[
v_{out}(s)\left(1 - \left(gm - \frac{G(s)}{R_i}\right)\right)\left(R_{load} \parallel \left(r_C + \frac{1}{sC_{out}}\right)\right) = -i_{out}(s)\left(R_{load} \parallel \left(r_C + \frac{1}{sC_{out}}\right)\right)
\]
Compensating the Buck – Method 3

- The output impedance is thus:

\[ Z_{out,CL}(s) = \frac{\hat{v}_{out}(s)}{\hat{i}_{out}(s)} = \frac{R_{load} \parallel \left( r_C + \frac{1}{sC_{out}} \right)}{1 - \left( \frac{gm - \frac{G(s)}{R_i}}{1} \right) \left( R_{load} \parallel \left( r_C + \frac{1}{sC_{out}} \right) \right)} \]

- Giving a small massage, we obtain:

\[ Z_{out,CL}(s) = \frac{R_{load} R_i}{R_i + G(s) R_{load} - R_i R_{load} \cdot gm} \cdot \frac{1 + s r_C C_{out}}{1 + s C_{out} \left( \frac{R_i R_{load} + R_i r_C + G(s) R_{load} r_C - R_i R_{load} r_C \cdot gm}{R_i + G(s) R_{load} - R_i R_{load} \cdot gm} \right)} \]

*There is a massage for you*
Compensating the Buck – Method 3

Factoring and re-arranging, we have:

\[ Z_{out,CL}(s) = R(s) \frac{1 + s/\omega_{z1}}{1 + s/\omega_{p1}} \quad R(s) = \frac{R_{load} R_i}{R_i + G(s) R_{load} - R_i R_{load} \cdot \text{gm}} \quad \omega_{z1} = \frac{1}{r_C C_{out}} \]

\[ \omega_{p1} = \frac{1}{C_{out} \left( \frac{R_i R_{load} + R_i r_C + G(s) R_{load} r_C - R_i R_{load} r_C \text{gm}}{R_i + G(s) R_{load} - R_i R_{load} \cdot \text{gm}} \right)} \]

Now make \( Z_{out,CL}(s) = r_C \) and extract \( G(s) \):

\[ G(s) = \frac{R_i \left( R_{load} - r_C + R_{load} r_C \text{gm} \right)}{r_C R_{load}} \frac{1 + s C_{out} \frac{R_{load} r_C^{2} \text{gm} - r_C^{2}}{R_{load} r_C^{2} \text{gm} + r_C^{2}}}{1 + s r_C C_{out}} \]
Compensating the Buck – Method 3

- The compensator brings a single pole/zero response

\[ G(s) = G_0 \frac{1 + s/\omega_z}{1 + s/\omega_p} \]

\[ G_0 = \frac{R_s (R_{load} - r_C + R_{load} r_C g_m)}{r_C R_{load}} \approx \frac{R_s}{r_C} \]

\[ \omega_z = \frac{1}{C_{out} \frac{R_{load} r_C^2 g_m - r_C^2}{R_{load} - r_C + R_{load} r_C g_m}} \approx \frac{1}{C_{out} \frac{-r_C^2}{R_{load}}} \]

\[ \omega_p = \frac{1}{C_{out} r_C} \]

Very high frequency can be neglected

[dB]

\[ R_s = 0.6 \Omega \]
\[ r_C = 30 \text{ m}\Omega \]
\[ C_{out} = 220 \mu\text{F} \]
Compensating the Buck – Method 3

- Sub-harmonic oscillations at $\frac{F_{sw}}{2}$ can cause peaking
- Place a zero at $\pi F_{sw}$ as recommended by YAO and al.

\[
\frac{v_c(s)}{V_{out}(s)} = -\frac{R_1}{Z_1}
\]

\[
Z_1 = \frac{R_2 \left( R_3 + \frac{1}{sC_1} \right)}{R_2 + \left( R_3 + \frac{1}{sC_1} \right)}
\]

\[
G(s) = -\frac{R_2}{R_1} \left[ \frac{1 + sC_1(R_2 + R_3)}{1 + sR_3C_1} \right]
\]

YAO et al., "Design Considerations for VRM Transient Response Based on the Output Impedance", IEEE Proceedings, 2003
Compensating the Buck – Method 3

- Run the SPICE simulation with the CM PWM switch model

Parameters:
- \( Vin = 10.18 \) V
- \( se = 20k \) V
- \( Fsw = 100k \) Hz
- \( pi = 3.14159 \)
- \( R = 0.6 \)
- \( Cout = 220u \)
- \( wp = 1/(rC*Cout) \)

\[ Wz1 = pi * Fsw \]
\[ Wp1 = 1/(rC*Cout) \]
\[ G = R / rC \]

### Automated calculations

- \( R1 = 10k \)
- \( R2 = R1 / G \)
- \( R3 = R1 * Wz1 / (G * (Wp1 - Wz1)) \)
- \( C1 = (G * (Wp1 - Wz1)) / (R1 * Wp1 * Wz1) \)
Compensating the Buck – Method 3

- The output impedance is resistive, the system is stable.

\[ |Z_{out,CL}(s)| \]
\[ \angle Z_{out,CL}(s) \]

\[ |T_{OL}(s)| \]
\[ \angle T_{OL}(s) \]

\[ \varphi_m = 66^\circ \]
\[ f_c = 40\,\text{kHz} \]
Compensating the Buck – Method 3

- The output response is a square signal as expected

\[ \Delta V_{\text{out}} = 30 \text{ mV} \]

\[ \Delta I_{\text{out}} = 1 \text{ A} \]

\[ f_{Z_1} @ F_{SW} \]

\[ V_{\text{out}}(t) \]
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**Quality Factor and Phase Margin**

- What is Delay Margin?
- Gain Margin is not Enough
Output Impedance and Quality Factor

- How to get $Q$ and $\varphi_m$ from the available signals?

You need enough signal and ringing to extract the data.

Must remain small-signal!

- $Q = 3.6$

- $V_{\text{peak}1} = 8.21 \text{ V}$

- $V_{\text{peak}2} = 6.33 \text{ V}$

- $k = \frac{6.33 - 5}{8.21 - 5} = 0.414$

- $t_d = 841.5 \mu\text{s}$

- $Q = \sqrt{\left(\frac{\pi}{\ln k}\right)^2 + \frac{1}{4}}$

- $\zeta = \sqrt{\left(\frac{2\pi}{\ln k}\right)^2 + 1}$

- $f_0 = \frac{1}{t_d \sqrt{1 - \zeta^2}}$
Output Impedance and Quality Factor

- Extraction is difficult with flat ac 2\textsuperscript{nd}-order responses

5-V step response

$V_{\text{out}}(t)$

Ac analysis

$|H(s)|$

$\angle H(s)$
Output Impedance and Quality Factor

The phase drops at a different pace as $Q$ changes

The plots show $|H(s)|$ and $\angle H(s)$ for different values of $Q$: $Q = 10$, $Q = 5$, $Q = 3$, $Q = 1$, and $Q = 0.6$.
Output Impedance and Quality Factor

- Is there a link between $Q$ and the phase rate of change?

Group delay

$$\tau_g = -\frac{d\varphi(\omega)}{d\omega} \rightarrow [s]$$

- Let's apply the definition to a 2\textsuperscript{nd}-order network:

$$H(s) = \frac{1}{1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2} \quad s = j\omega$$

$$H(j\omega) = \frac{1}{1 - \frac{\omega^2}{\omega_0^2} + j\frac{\omega}{\omega_0 Q}}$$

$$a$$

$$b$$

$$|H(\omega)| = \sqrt{a^2 + b^2} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{\omega}{\omega_0 Q}\right)^2}}$$

$$\angle H(\omega) = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left[\frac{\frac{\omega}{\omega_0 Q}}{1 - \frac{\omega^2}{\omega_0^2}}\right]$$
Output Impedance and Quality Factor

- We can apply \( \tau_g \) definition to the argument

\[
\tau_g = -\frac{d \tan^{-1} \left( \frac{\omega}{\omega_b Q \left( 1 - \frac{\omega^2}{\omega_0^2} \right)} \right)}{d\omega} = \frac{Q\omega_b \left( \omega^2 + \omega_0^2 \right)}{Q^2 \omega^4 - 2Q^2 \omega^2 \omega_0^2 + Q^2 \omega_0^4 + \omega^2 \omega_0^4}
\]

\[
\tau_g = \frac{2Q}{\omega_0}
\]

- At \( \omega_0 \) the following formula links \( Q \) to the group delay

\[
Q = \frac{\tau_g \omega_0}{2} = \tau_g \pi f_0
\]
Output Impedance and Quality Factor

Let's apply the theory to a classical case, the $RLC$ filter

- $V_{p} = 5$
- $f_{0} = 1.2k$
- $L = 10u$
- $C = \frac{1}{(4\times3.14159^2\times f_{0}^2 \times L)}$
- $w_{0} = (L \times C)^{-0.5}$
- $Q = 0.6$
- $R = L \times w_{0} / Q$
- $R_{2} = 1/(Q \times C \times w_{0})$
- $Q_{1} = (\sqrt{L} / C) / R$
- $D_{\text{zeta}} = (R / 2) \times \sqrt{C / L}$
- $D_{\text{zeta}}1 = R / (2 \times L \times w_{0})$
- $Q_{3} = 1 / (2 \times D_{\text{zeta}})$
- $\text{per} = 1 / (f_{0} \times \sqrt{1 - D_{\text{zeta}}^2})$
- $t_{p} = 1 / (2 \times f_{0} \times \sqrt{1 - D_{\text{zeta}}^2})$

- plot the ac response
- calculate the group delay
- see if we can find $Q$
\( Z_{\text{out}}(s) \)

\[ f_0 = 1207 \text{ Hz} \]

\[ Q = \tau_g \pi f_0 = 1207 \times 158 \mu s \times 3.14159 = 0.599 \]
Output Impedance and Quality Factor

Knowing $Q$ can also lead us to the phase margin $\phi_m$.

$T_{OL}(s) = \frac{1}{s \omega_0 \left(1 + \frac{s}{\omega_2}\right)}$
Output Impedance and Quality Factor

- If we consider the open-loop gain around \( f_c \) only...

\[
T_{OL}(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{s}{\omega_c}\right)}
\]

- Closed-loop

\[
T_{OL}(s) = \frac{1}{1 + T_{OL}(s)} = \frac{1}{\frac{s^2}{\omega_0\omega_c} + \frac{s}{\omega_0} + 1}
\]

- Unity return

\[
\frac{1}{\omega_0\omega_c} + \frac{s}{\omega_0} = \frac{1}{\omega_c + \frac{s}{\omega_0} + 1}
\]

- Identify

\[
Q_c = \sqrt{\frac{\omega_0}{\omega_c}} \quad \omega_c = \sqrt{\omega_0\omega_c}
\]

- We want to link \( Q_c \) and the crossover frequency

\[
T_{OL}(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{s}{\omega_c}\right)}
\]

\[
\omega_0 = Q_c^2\omega_2
\]

\[
T_{OL}(s) = \frac{1}{\left(\frac{s}{Q_c\omega_2}\right)\left(1 + \frac{s}{\omega_2}\right)}
\]
Output Impedance and Quality Factor

- Calculate the $T_{OL}$ magnitude at crossover with $Q_c$ in:

  \[
  \left| \frac{1}{\frac{j \omega_c}{Q_c^2 \omega_2}} \right| = \frac{Q_c^2 \omega_2^2}{\sqrt{\omega_c^2 \omega_2^2 + \omega_c^4}}
  \]

  Solve $\omega_c$

  \[
  \omega_c = \frac{\omega_2 \sqrt{\sqrt{1 + 4Q_c^4} - 1}}{\sqrt{2}}
  \]

- Derive the argument of $T_{OL}$, simplify it:

  \[
  \arg T_{OL}(s) = \arg \left( \frac{1}{\frac{s}{\omega_0} + \frac{s}{\omega_2}} \right) = \arg \left( \frac{j Q_c^2 \omega_2^2}{- \omega_c \omega_2 - j \omega_c^2} \right) = \arg \left( - \frac{1}{-j \frac{\omega_c \omega_2}{Q_c^2 \omega_2^2} + \frac{\omega_c^2}{Q_c^2 \omega_2^2}} \right)
  \]

  \[
  \arg T_{OL}(s) = \arg(-1) - \tan^{-1} \left( \frac{\omega_c \omega_2}{Q_c^2 \omega_2^2} \right) = -\pi - \tan^{-1} \left( \frac{-\omega_c}{\omega_2} \right)
  \]
Output Impedance and Quality Factor

- If we substitute $\omega_c$ by its definition, we have:

$$\arg T_{OL}(s) = -\pi - \tan^{-1}\left(-\frac{2}{\sqrt[4]{1 + 4Q_c^4} - 1}\right) = -\pi + \tan^{-1}\left(-\frac{2}{\sqrt[4]{1 + 4Q_c^4} - 1}\right)$$

$$\angle T_{OL}(f_c) - \varphi_m = -\pi$$

$$\varphi_m = \angle T_{OL}(f_c) + \pi$$

$$\varphi_m = \tan^{-1}\left(-\frac{2}{\sqrt[4]{1 + 4Q_c^4} - 1}\right)$$
Output Impedance and Quality Factor

- We can now extract the closed-loop quality coefficient:

\[ Q_c = \frac{\sqrt{1 + \tan(\varphi_m)^2}}{\tan(\varphi_m)} = \frac{\cos(\varphi_m)}{\sin(\varphi_m)} \]

\[ \varphi_m = \cos^{-1}\left(\frac{\sqrt{4Q_c^4 + 1} - 1}{2Q_c^2}\right) \]

![Graph showing Q_c and \(\varphi_m\) vs. frequency]

Closed-loop gain

- \(T(s)\) for different phase angles:
  - \(\varphi_m = 10^\circ\)
  - \(\varphi_m = 20^\circ\)
  - \(\varphi_m = 30^\circ\)
  - \(\varphi_m = 45^\circ\)
  - \(\varphi_m = 90^\circ\)
Output Impedance and Quality Factor

- The formula considers the vicinity of $f_c$ only: precision?

\[ Q_c = \sqrt{\left(\frac{\pi}{\ln\left(\frac{47}{128}\right)}\right)^2 + \frac{1}{4}} = 3.1 \]

\[ \varphi_m = \cos^{-1}\left(\frac{\sqrt{4Q_c^4 + 1 - 1}}{2Q_c^2}\right) \approx 18^\circ \]

"Revisiting the Response of Closed Loop of PWM Converters ", S. Ben-Yaakov, IEEE Apec 2008
Output Impedance and Quality Factor

We can ac sweep the output impedance also and check $\tau_g$

\[ Q = \tau_g \pi f_0 = 3.5k \times 253u \times 3.14159 \approx 2.8 \quad \Rightarrow \quad \phi_m \approx 20.4^\circ \]
Output Impedance and Quality Factor

- We can apply the technique to simple linear regulators

- You have no means to perform open-loop analysis

- Plot its output impedance with a network analyzer…

Output Impedance and Quality Factor

- Plot magnitude and phase of $Z_{out}$ and compute $\tau_g$

$$Q = \frac{\tau_g \pi f_0}{1k \times 574.9u \times 3.14159} = 1.8$$

$$\varphi_m = \cos^{-1} \left( \frac{\sqrt{4Q^4 + 1} - 1}{2Q^2} \right) = \cos^{-1} \left( \frac{\sqrt{4 \times 1.8^4 + 1} - 1}{2 \times 1.8^2} \right) = \cos^{-1} (857m) \approx 31^\circ$$
Output Impedance and Quality Factor

- The LDO circuit is made of the following elements

- We can open its loop and plot the open-loop gain $T_{OL}(s)$
The open-loop plot confirms the experimental results.
Output Impedance and Quality Factor

The phase margin is increased to 60°

\[ Q = \tau_g \pi f_0 = 1.3k \times 185.5u \times 3.14159 \approx 0.76 \]

\[ \varphi_m = \cos^{-1}\left(\frac{\sqrt{4Q^4 + 1} - 1}{2Q^2}\right) = \cos^{-1}\left(\frac{\sqrt{4 \times 0.76^4 + 1} - 1}{2 \times 0.76^2}\right) = \cos^{-1}(457m) \approx 63° \]
Course Agenda

- Introduction to Control Systems
- Shaping the Error Signal
- How to Implement the PID Block?
- The PID at Work with a Buck Converter
- Considering the Output Impedance
- Classical Poles/Zeros Placement
- Shaping the Output Impedance
- Quality Factor and Phase Margin
- What is Delay Margin?
- Gain Margin is not Enough
Considering a Delay in the Loop

- Before a decision is actually executed, a delay occurs.
- The delay can be digital (computation time) or analogue.
- A typical delay is the duty-ratio conversion in a VM converter.

Prop. delay in the 50 – 100 ns range.
A Delay is a Time-Domain Shift

The output signal is the input signal that occurred \( \tau \) s before.

\[
y(t) = u(t - \tau)
\]
Deriving the Delay

To account for the delay, we need its Laplace expression

\[ \mathcal{L}[\text{delay}] = ? \]
Deriving the Delay

To account for the delay, we need its Laplace expression

\[ \mathcal{L}[y(t)] = \mathcal{L}[u(t - \tau)] \]

Let's start with a sinewave phasor expression

\[ u(t) = Ae^{j\omega t} \]

\[ y(t) = Ae^{j\omega(t-\tau)} = Ae^{j\omega t}e^{-j\omega \tau} = u(t)e^{-j\omega \tau} \]
Deriving the Delay

- Let's take the Laplace transform of the new expression

\[ Y(s) = U(s)e^{-st} \]

\[ \frac{Y(s)}{U(s)} = e^{-st} \]

\[ \frac{Y(j\omega)}{U(j\omega)} = e^{-j\omega t} \]

- Euler phasor formula uses the argument as the exponent

\[ e^{-j\omega t} \rightarrow e^{j\varphi} \]

\[ \arg e^{-j\omega t} = -\omega t \]

\[ |e^{-j\omega t}| = 1 \]

- A delay block is a simple delay line!

\[ u(t) \xrightarrow{\text{UTD}} y(t) \]

K = 1, \( t_d = 250 \text{ ns} \)
Building the Delay

- A delay line can time-shift the input signal

- Best simulation practice is to buffer the input and the output


Parameters:

\[ \tau = 250 \text{n} \]

Delay block:

\[ T_1 \quad T_D = \{ \tau \} \]

\[ 50 = Z_0 \]

\[ + \text{Vin} \quad AC = 1 \]

\[ \quad 3 \]

\[ R1 \quad 50 \]

\[ \quad 2 \quad \text{Vout} \]
Building the Delay

- A Bode plot confirms the 0-dB magnitude over frequency

\[ |e^{-\pi\tau}| \]

**Magnitude is flat**

\[ \angle e^{-\pi\tau} \]

\[ \varphi_{100\text{kHz}} = -\omega\tau = -100k \times 6.28 \times 250n \times \frac{180}{\pi} = -9^\circ \]
Adding the Delay in the Laplace Domain

We can now update our transmission chain with the delay

\[ H(s) = \frac{e^{-st}}{V_{peak}} \frac{1 + s/\omega_1}{1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2} \]
Adding the Delay in the Laplace Domain

- How do you deal with the term $e^{-st}$ in the transfer function?
- You don't: replace it with a poles/zeros combination!
  - A pole will bring phase shift as frequency increases
    \[
    \frac{1}{1 + s/\omega_r} \quad \Rightarrow \quad \arg = -\tan^{-1}\left(\frac{\omega}{\omega_r}\right)
    \]
- But you still need to compensate the transmittance decrease
  - A zero will do but now, all is neutralized!
    \[
    \left\{\frac{1 + s/\omega_r}{1 + s/\omega_r}\right\} \quad \Rightarrow \quad \text{mag} = \frac{\sqrt{1 + \left(\frac{\omega}{\omega_r}\right)^2}}{\sqrt{1 + \left(\frac{\omega}{\omega_r}\right)^2}} = 1
    \]
    \[
    \arg = \tan^{-1}\left(\frac{\omega}{\omega_r}\right) - \tan^{-1}\left(\frac{\omega}{\omega_r}\right) = 0
    \]
Calling the RHP Zero for Help

- Replacing the LHP zero by a RHP zero does the job
- Both pole and zero magnitude neutralize each other
- The RHPZ phase lags and cumulates with that of the pole

\[ e^{-st} \approx \frac{1 - \frac{s}{\omega_r}}{1 + \frac{s}{\omega_r}} \]
Mapping the Delay to the Pole/Zero Position

- Both arguments must be equal:

\[ \arg(e^{-s\tau}) = \arg\left(\frac{1-s/\omega_r}{1+s/\omega_r}\right) \]

\[ -\omega\tau = \arg(1-s/\omega_r) - \arg(1+s/\omega_r) \]

- Replacing \( s \) by \( j\omega \):

\[ -\omega\tau = \tan^{-1}\left(-\frac{\omega}{\omega_r}\right) - \tan^{-1}\left(\frac{\omega}{\omega_r}\right) \approx -2\tan^{-1}\left(\frac{\omega}{\omega_r}\right) \]

- Use the arctangent Taylor series equivalent:

\[ \tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots \]

\[ -\omega\tau \approx -2\left[\frac{\omega}{\omega_r} - \frac{\omega^3}{3\omega_r^3} + \frac{\omega^5}{5\omega_r^5}\right] \approx 0 \]
We Have the Padé Approximation

- Solving for $\omega \tau$ gives us...
  
  $$\omega \tau = \frac{2}{\tau}$$

- Substituting $\omega \tau$ in our first expression
  
  $$e^{-s\tau} \approx \frac{1 - \frac{s\tau}{2}}{1 + \frac{s\tau}{2}}$$

- This is the 1st-order Padé approximant of an exponential

  $$e^x \approx \frac{1 + \frac{x}{2}}{1 - \frac{x}{2}} \quad \text{2nd-order} \quad e^x \approx \frac{1 + \frac{1}{2}x + \frac{1}{12}x^2}{1 - \frac{1}{2}x + \frac{1}{12}x^2}$$

Henri Padé
1863-1953

http://en.wikipedia.org/wiki/Padé_approximant
Padé Approximant Frequency Response

- When frequency increases, a phase deviation occurs

```
\begin{align*}
\left| e^{-s\tau} \right| &= \frac{1 - s\tau/2}{1 + s\tau/2} \\
\text{arg} \left( \frac{1 - s\tau/2}{1 + s\tau/2} \right) &= e^{-s\tau}
\end{align*}
```

Use higher order approximants to improve precision.
How to Avoid the Delay Line?

- A delay line adds computational burden in simulations
- Is there any simpler circuit that could be used?

\[ V_{out}(s) = V_1(s) - V_1(s) \frac{R_i}{1/sC_i} = V_1(s)[1 - sR_iC_i] \]
How to Avoid the Delay Line?

- Good phase response of the analogue circuit

![Graph showing phase and magnitude responses for op amp and delay line]
Delay Margin versus Phase Margin

- The characteristic equation is affected by the delay:
  \[
  \chi(s) = 1 + T(s) = 0 \quad \rightarrow \quad \chi(s) = 1 + e^{-s\tau}T(s) = 0
  \]

Unity return denominator \( T(s) \)

- The stability condition for the magnitude does not change:
  \[
  |e^{-s\tau_{\text{max}}} T(\omega_c)| = 1 \quad \rightarrow \quad |e^{-s\tau}| = 1 \quad \rightarrow \quad |T(\omega_c)| = 1
  \]

- The stability condition for the argument does change:
  \[
  -\pi = \arg(e^{-s\tau_{\text{max}}}) + \arg T(\omega_c) = -\omega\tau_{\text{max}} + \arg T(\omega_c)
  \]

The max acceptable delay in the loop

\[
\tau_{\text{max}} = \frac{\phi_m}{\omega_c}
\]

Solving for \( \tau_{\text{max}} \)

\[
\phi_m = \pi + \arg T(\omega_c)
\]
Delay Margin versus Phase Margin

1. The delay margin is thus defined by:

\[ \Delta \tau = \tau_{\text{max}} - \tau \]

- Current delay
- Maximum acceptable delay

1. A buck converter features a 250-ns internal delay
2. At a 100-kHz crossover, the phase margin is 49.5°
3. The maximum delay, accounting for 250 ns, is:

\[ \tau_{\text{max}} = \frac{\phi_m}{\omega_c} = \frac{49.5}{2\pi \times 100k \times 180} = 1.375 \, \mu s \]

\[ \Delta_r = 1.375 - 0.250 = 1.125 \, \mu s \]

Maximum acceptable extra delay
Checking via Simulation

- The delay is simply added in series with the PWM block

Delay built with an adapted transmission line
Checking via Simulation

With a 250-ns delay, the phase margin is acceptable.
Checking via Simulation

- If we add 1.125 µs to 250 ns, no phase margin at all!
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Gain Margin Defines the Robustness

- GM defines the robustness of a system to gain variations

**Conditions for oscillations:**

\[ \angle T_{OL}(s) = 0^\circ \]
\[ |T_{OL}(s)| = 0 \text{ dB} \]

- If the gain drifts up by 38 dB, we have oscillations
Gain Margin Defines the Robustness

- In Bode representation but also in Nyquist

\[ \angle T(s) = \pi \text{ or } -\pi \]

\[ \text{Im} T(s) = 0 \]

\[ \text{GM} = \frac{1}{\sqrt{\text{Re} T(f_+)^2 + \text{Im} T(f_+)^2}} \]

\[ \frac{1}{\text{GM}} = \text{Re} T(f_+) \]
Gain Margin in Nyquist

- What really matters is the distance to the "-1" point

\[ h = \sqrt{\left(-\left(1 - \text{Re}(\omega)\right)\right)^2 + \text{Im}(\omega)^2} \]

\[ h = |1 + T(s)| \]
A closed-loop system rejects the incoming perturbation $u_1$.

\[ y(s) = u_2(s) \frac{T(s)}{1+T(s)} - u_1(s) \frac{1}{1+T(s)} \]

Sensitivity function $S$

\[ h = |1 + T(s)| = \frac{1}{S} \]
Watch for a Peak in $S$

- The trajectory must keep away from the "-1" point
- A 0.5-radius circle detects a peak in $S$: modulus margin

![Diagram showing peak detection and margin violation](image-url)
Bode can also Show the Modulus Margin

\[ |S| = \frac{1}{1 + T(s)} \]

-800m -400m 0 400m 800m
real

-800m -400m 0 400m 800m
imag

\[ \omega \]

\[ \Delta M \]

\[ \frac{1}{S_{\text{min}}} \]

\[ \frac{1}{GM} \]

6-dB limit

\[ dB \]

\[ \text{frequency} = 26.3 \text{ kHz} \]

\[ \text{real} = -649 \text{m} \]

\[ \text{imag} = -111 \text{m} \]

\[ \text{sens} = 8.67 \text{ dB} \]
Conclusion

- Switching or linear power supplies are regulators
- Applying a pure mathematical compensation brings problems
- Engineering judgment found output impedance guilty
- Lack of sufficient gain at resonance brings oscillations
- Standard poles/zeros placement method gives good results
- For high-speed dc-dc converters, resistive shaping rules
- $Q$ to phase margin approximation requires engineering judgment
- Less known delay and modulus margins are useful figures!

Merci !
Thank you!
Xiè-xie!