The Right–Half –Plane Zero, a Two-Way Control Path

Christophe BASSO – ON Semiconductor
14, rue Paul Mesplé – BP53512 - 31035 TOULOUSE Cedex 1 - France

The small-signal analysis of power converters reveals the presence of poles and zeros in the transfer functions of interest, e.g. the control to the output variable. The zeros occur in the numerator of the expression whereas the poles are located in the denominator. The stability analysis of the transfer function consists in looking at the position these poles and zeros occupy in the s-plane. For a stable converter, one condition is that both the zeros and the poles reside in the left half portion of the plane: we are talking about negative roots. For a pole, a position in the left plane implies an exponentially decaying temporal response, hence asymptotically stable. To the opposite, when placed on the right side in the s-plane, a step response will lead to a diverging response as the associated exponential term exhibits a positive exponent: this is a positive root. For some converter architectures, a zero can sometimes be the positive root to the numerator of the control-to-output transfer function. How can this happen and what are the consequences of such a positive zero also called a Right–Half-Plane Zero (RHPZ), this is the object of the present paper.

A two-step conversion process

Figure 1 represents a classical boost converter where two switches appear. A power switch *SW*, usually a MOSFET, and a diode *D*, sometimes called a catch diode. In the Continuous Conduction Mode of operation (CCM), the inductor current I_L flows in the power switch *SW* during the on time or DT_{sw}. During the off time, or $(1\text{-}D)T_{sw}$, the power switch is open and the inductor current goes to the output diode, further feeding an output network made of the capacitor and the load. Regardless of the control method, voltage or current-mode, this configuration assumes that energy is first stored in the inductor during the on time and then transferred to the output during the off time.

In Figure 2 appears an equivalent representation of the boost converter where the switch/diode network has been replaced by a single pole double throw switch which alternatively routes the inductor current in the two different branches: the power switch or the output diode. If a designer would observe the currents circulating in the output diode, he would see Figure 3 typical waveforms. Our boost converter is designed to deliver power to a given load. The variable of interest, in our case, is thus the available output current I_{out}. This current is actually made of a dc portion on which is superimposed a switching ripple. In theory, the ripple goes into the capacitor and the dc current circulates in the load. The dc current delivered by the boost converter is nothing else than the diode average current I_d. Mathematically, this current can be expressed by:

$$I_{out} = I_d = I_L\left(1-D\right) \tag{1}$$

where I_d is the average diode current also equal to the dc output current I_{out} and D is the duty cycle.

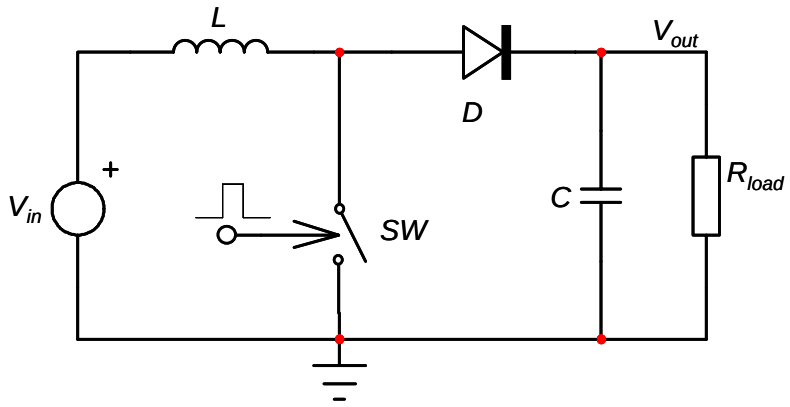

Figure 1: a boost converter features two switches

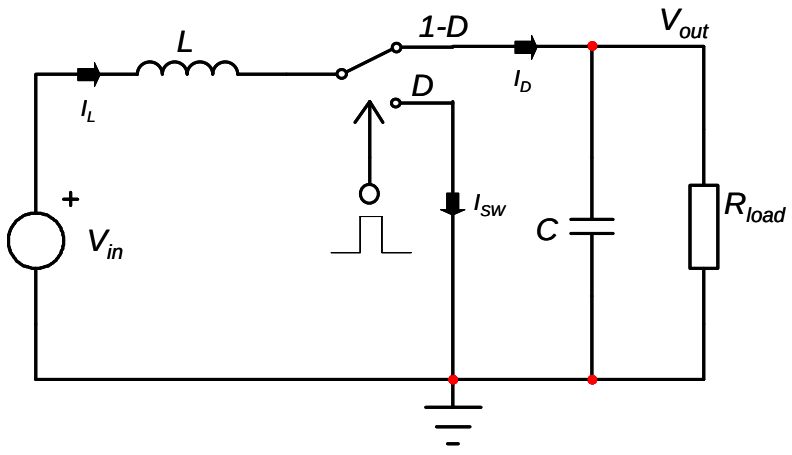

Figure 2: a single pole double throw switch represents the combined operations of the diode and the power switch.

The diode current

In Figure 3, on the left side, we can see the current in the diode jumping to the peak inductor current as soon as the switch opens. Then, the current decays with a slope imposed by the voltage across the inductor during the off time. The diode average current in the left picture is I_{d0} and obeys Eq. (1). Now, on the right picture, the duty cycle has slightly increased. The inductor current peaks a little higher but, given the reduction of the $(1-D)$ term in Eq. (1) due to the increase of *D*, the average current I_{d1} is lower than before.

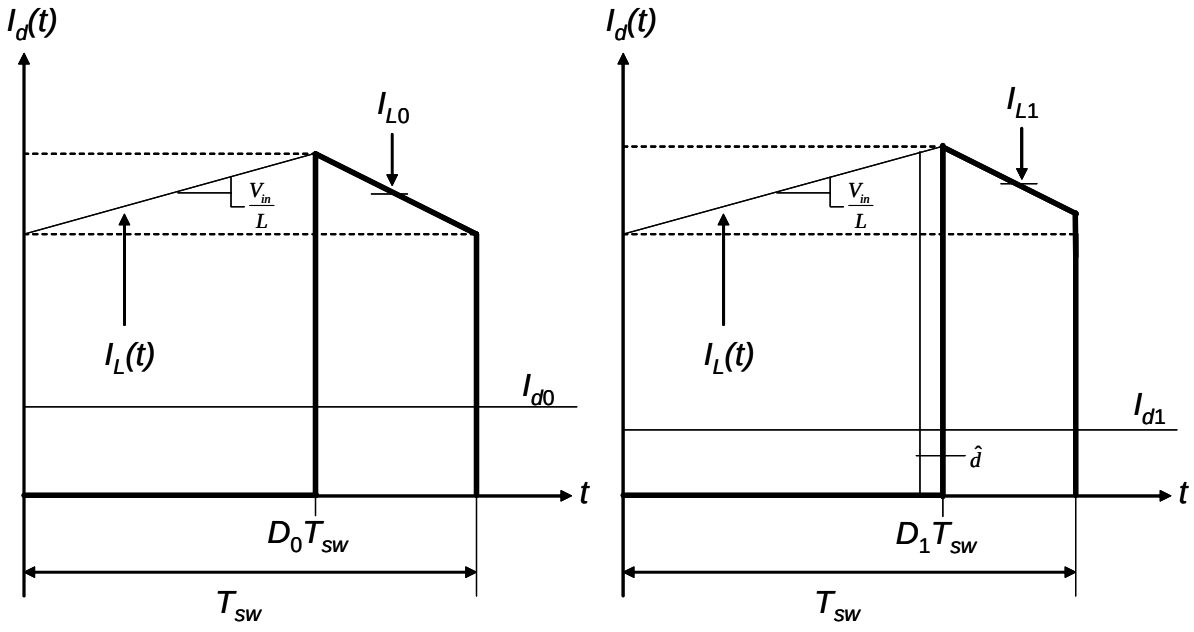

Figure 3: the current observed in the output diode with two different duty cycles.

As one can see from Eq. (1), if *D* suddenly increases to correct a perturbation, then, to let I_{out} follow-up, we need an immediate increase in the inductor current I_L as well. The problem relates to the average inductor current which is limited in slew rate. If the inductor average current change $\frac{dI_L}{dt}$ is slower than the duty cycle change $\frac{dD}{dt}$, then the output current I_{out} goes down immediately until the inductor current builds-up and eventually catches-up with the set point imposed by the loop. However, if I_{out} goes down, so does V_{out}, immediately sensed by the feedback loop. The controller increases the duty cycle and sees a decrease in the

voltage, the reverse of what the loop polarity is supposed to be: this is the physical effect of the right-half-plane zero located in the control-to-output transfer function.

What is the pace at which the average inductor current can change? Lenz's law instructs us that the instantaneous current change rate in an inductor obeys the following formula:

$$\frac{dI_L(t)}{dt} = \frac{V_L(t)}{L} \tag{2}$$

On average, it simply follows:

$$\frac{dI_L}{dt} = \frac{V_L}{L} \tag{3}$$

where I_L and V_L respectively represent the average inductor current and voltage values. The exercise now consists in calculating the average value across our inductor. By considering the weighted period of time during which V_{in} or $V_{out}–V_{in}$ are applied across L, we have:

$$V_L = V_{in}D - (V_{out} - V_{in})(1-D) = V_{out}(D-1) + V_{in} \tag{4}$$

Let's assume the following boost operating parameters:

V_{in} = 10 V
V_{out} = 24 V
D_0 = 0.583
R_{load} = 240 Ω
L = 1 mH

With a 58.3% duty cycle, the converter delivers 24 V. Now suppose that the duty cycle jumps to D_1 = 59% or a difference of 0.7%. What is the inductor average current slope in this case? Considering a large output capacitor, the output voltage stays constant during the duty cycle change. Applying Eq. (4) gives a transient average inductor voltage of:

$$V_L = V_{out}(D_1 - 1) + V_{in} = 24 \times (0.59 - 1) + 10 = 160\ mV \tag{5}$$

Back to Eq. (3), the maximum average current slope is therefore:

$$\frac{dI_L}{dt} = \frac{V_L}{L} = \frac{160m}{1m} = 160\ \mu A/\mu s \tag{6}$$

a rather low value.

When the duty cycle changes from 58.3% to 59%, it implies an output voltage change of:

$$V_{out} = \frac{V_{in}}{1-D} = \frac{10}{1-0.589} = 24.33\,V \tag{7}$$

With a constant 240-Ω load, the output current will increase to:

$$I_{out} = \frac{V_{out}}{R_{load}} = \frac{24.33}{240} = 101.37\ mA \tag{8}$$

Brought back to the inductor change, the output current variation given by Eq. (8) must be accompanied by an average inductor current variation of:

$$\Delta I_L = \frac{V_{in}}{R_{load}}\left[\frac{1}{(1-D_1)^2}-\frac{1}{(1-D_0)^2}\right]=\frac{10}{240}\left[\frac{1}{(1-0.589)^2}-\frac{1}{(1-0.583)^2}\right]=246.65-239.8=6.85\,mA \tag{9}$$

Given an average inductor slope 160 µA/µs, this current variation will only be possible within a timeframe of:

$$dt=\frac{6.85m}{160u}=42.8\,\mu s \tag{10}$$

If the duty cycle is swept from 58.3% to 59% in much less time than 42.8 µs, the inductor current will not build-up at a sufficient pace to make the output current rise at the same speed. As an immediate result, the output current drops rather than increases. On the contrary, if the duty cycle sweep is slow enough, the current can increase in the inductor at sufficient speed to compensate the reduction in (*1–D*): the output voltage goes up. This is the reason why a reduction in the available loop bandwidth naturally limits the duty cycle slew rate and gives time for the inductor current to build up.

An average model to visualize the effects

Figure 4 depicts a voltage mode boost converter modelled using the newly derived auto-toggling model Ref. [1] based on the PWM switch model. In this figure, we will sweep the duty cycle from 58.3% to 59% at different speeds and then observe the pertinent waveforms:

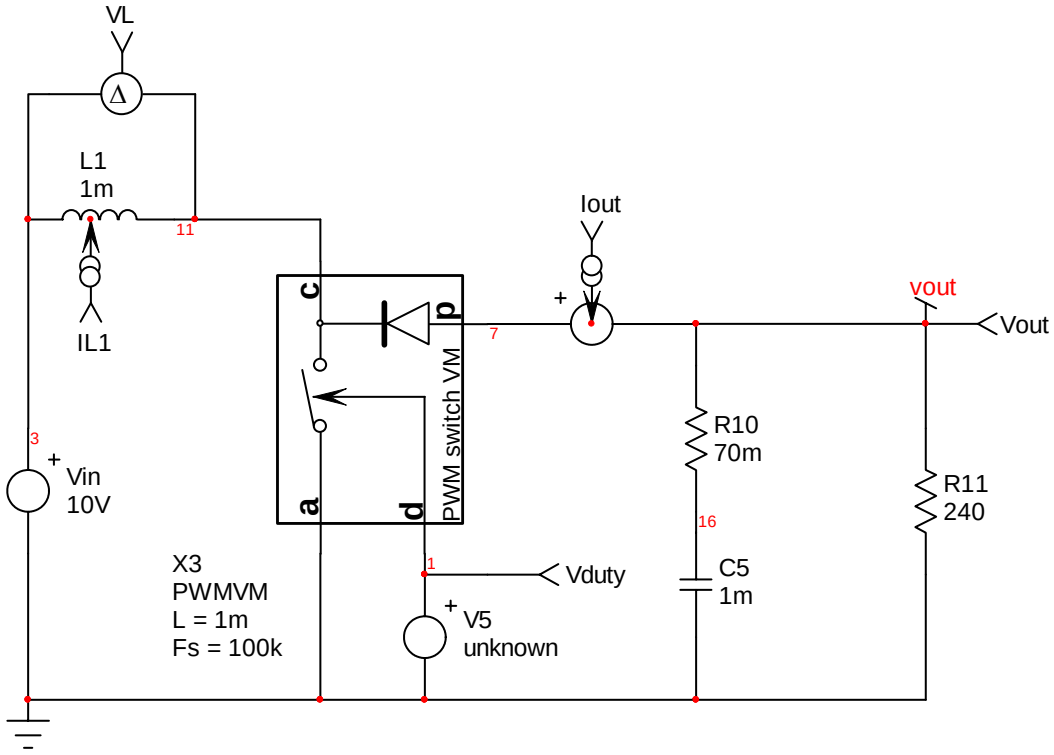

Figure 4: a boost converter using an average model is a good tool to observe the RHPZ effects.

The results appear in Figure 5 and Figure 6. In Figure 5, the duty-cycle is slowly swept in 200 µs and one can see that the output voltage rises up without any negative portion: the inductor current can keep up with the duty cycle change and the converter responds, in time, to the step. The situation differs in Figure 6 where the sweep time is reduced to 10 µs. In this particular case, the inductor average current cannot positively answer the required change and the output current drops. The same occurs in the output voltage and, if a voltage loop would be involved, an oscillation would take place.

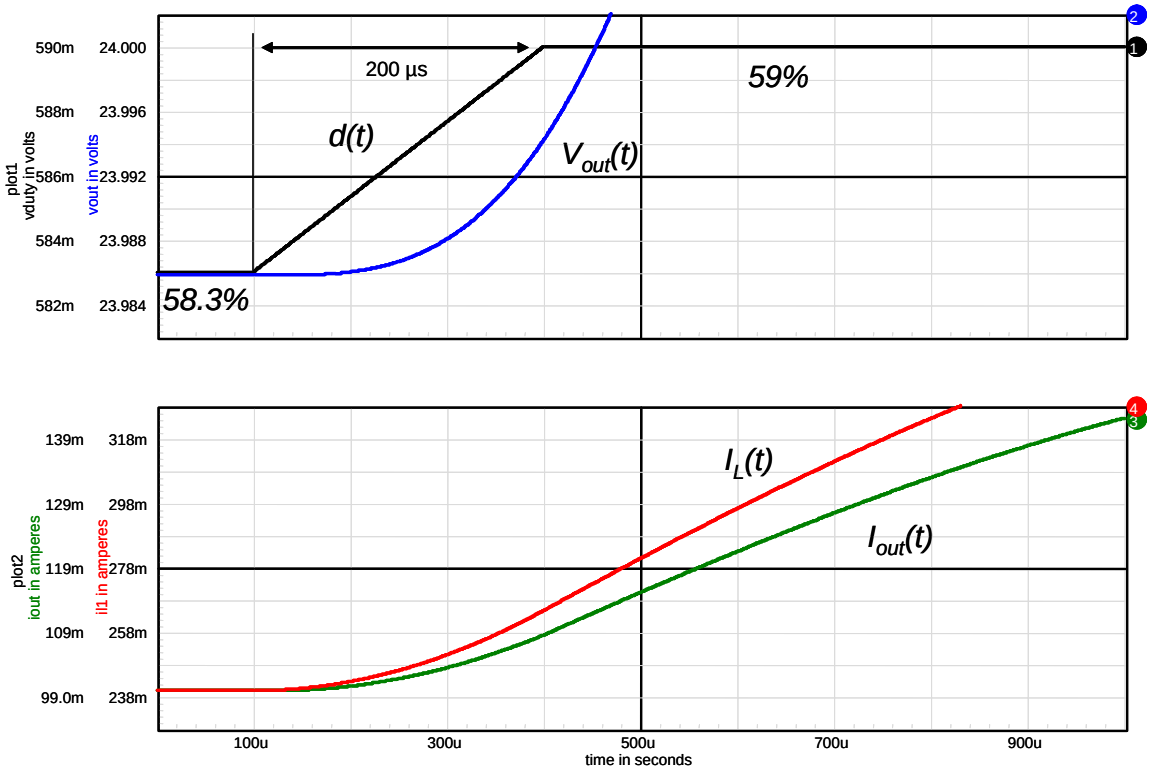

Figure 5: when the duty cycle is slowly swept, the output voltage stays positive and the control law works in the right direction.

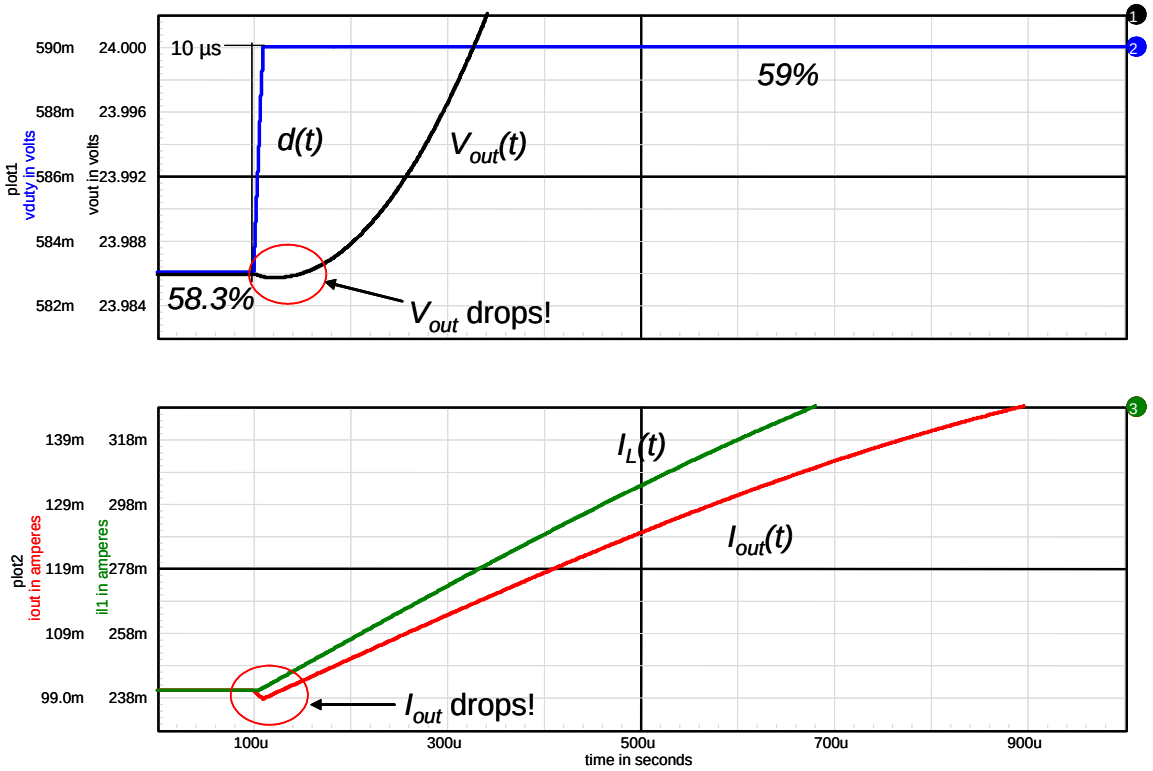

Figure 6: if the duty cycle is quickly swept, the average inductor current limits the output current slew rate and the output voltage drops until the current builds-up. During a short time, the control law is reversed!

Based on the above observations, we can state that:

- In CCM, the average inductor current is limited in slew rate by the available voltage during the duty cycle change. A large inductor worsens the situation, a small inductor improves it.
- If the duty cycle change imposed by the feedback loop tries to set an output current variation beyond the inductor slew rate capabilities, the output voltage drops and oscillations occurs. On the contrary, slower duty cycle changes will correctly propagate to the output without endangering the loop stability.
- As a preliminary conclusion, if we limit the duty cycle slew rate or simply truncate the available loop bandwidth, we have a means to fight the control-to-output RHPZ inherent to the CCM boost converter.

Now that we have physically observed the potential instability, let us try to develop a small-signal model.

Small signal study

Equation 1 represents a non-linear large signal expression. To deal with a small-signal ac equation in which poles and zeros could appear, we need to apply a linearization process around an operating point. There are two ways to do it:

1. Perturb all dc terms with a small ac modulation. That is to say, replace all terms susceptible to change by a static value plus an ac modulation:

$$I_{out} + \hat{i}_{out} = \left(I_L + \hat{i}_L\right)\left(1 - \left(D + \hat{d}\right)\right) \tag{11}$$

$$I_{out} + \hat{i}_{out} = I_L - \hat{d}I_L - DI_L + \hat{i}_L - \hat{d}\hat{i}_L - D\hat{i}_L \tag{12}$$

- Collect and sort dc terms and ac terms to form two different equations. Get rid of the ac cross products as they are of negligible contribution (small by small leads to a smaller result):

$$I_{out} = I_L - DI_L = I_L\left(1 - D\right) \tag{13}$$

$$\hat{i}_{out} = \hat{i}_L\left(1 - D\right) - \hat{d}I_L \tag{14}$$

- We now have a dc equation which gives us the bias point of our boost converter. The ac equation is the small-signal response of the output current to a perturbation in the inductor current and the duty cycle. This is the equation we are looking for.

2. The second method deals with partial derivative. In some cases, the individual variable perturbations can lead to complicated expressions where the final sort of dc and ac equations represents a tedious exercise. When the bias point is already known, it is faster to use partial derivatives. A partial derivative actually evaluates the sensitivity of a function to its individual variables. The result is then the ac equation we are looking for, without dc terms and without neglecting ac cross products.

- Applying the method to Eq. 1, we have:

$$\hat{i}_{out} = \left(\frac{\partial I_{out}}{\partial I_L}\hat{i}_L\right)_D + \left(\frac{\partial I_{out}}{\partial D}\hat{d}\right)_{I_L} = \hat{i}_L\left(1 - D\right) - \hat{d}I_L \tag{15}$$

In Eqs. (14) or (15), the ac inductor current $\hat{i}_L$ appears. What is the expression of an ac inductor current? Simply the ac inductor voltage divided by the inductor impedance. Let us find the expression of the ac inductor voltage by first deriving its average large signal expression, what we already did with Eq. (4). On average, when the converter is at the equilibrium, this equation gives zero. However, under an ac excitation, the average inductor voltage is also ac modulated across zero. By using the partial derivative option, we can see that the ac inductor voltage, in this case, is expressed by:

$$\hat{v}_L = \hat{v}_{out}\left(D-1\right) + \hat{d}V_{out} \tag{16}$$

From Eq. (4), we can see that the input term V_{in} has disappeared since the input voltage is considered constant during the ac analysis. Furthermore, if we consider a large output capacitor, its impedance at the ac excitation can be considered close to zero, helping to further simplify the expression to:

$$\hat{v}_L \approx \hat{d}V_{out} \tag{17}$$

Having the ac inductor voltage, it is easy to obtain the ac inductor current we are looking for:

$$\hat{i}_L\left(s\right) = \frac{\hat{v}_L\left(s\right)}{Z_L} = \frac{\hat{d}\left(s\right)V_{out}}{sL} \tag{18}$$

Substituting Eq. (18) into Eq. (15) gives the final ac output current expression:

$$\hat{i}_{out}\left(s\right) = \frac{\hat{d}\left(s\right)V_{out}}{sL}\left(1-D\right) - \hat{d}\left(s\right)I_L \tag{19}$$

The average inductor current I_L is the source current I_{in}. Considering a 100% efficiency power conversion, we can write:

$$V_{in}I_{in} = V_{out}I_{out} = \frac{{V_{out}}^2}{R} \tag{20}$$

From which we have:

$$I_{in} = I_L = \frac{{V_{out}}^2}{V_{in}R_{load}} = \frac{V_{out}}{V_{in}}\frac{V_{out}}{R_{load}} = \frac{V_{out}}{\left(1-D\right)R_{load}} \tag{21}$$

Substituting Eqs. (21) into Eq. (19), we obtain:

$$\frac{\hat{i}_{out}\left(s\right)}{\hat{d}\left(s\right)} = \frac{V_{out}D'}{sL} - \frac{V_{out}}{D'R_{load}} \tag{22}$$

Now factoring the first term and re-arranging, we have:

$$\frac{\hat{i}_{out}\left(s\right)}{\hat{d}\left(s\right)} = \frac{V_{out}D'}{sL}\left(1-\frac{sL}{D'^2R_{load}}\right) = \frac{\left(1-\dfrac{s}{\omega_{z_2}}\right)}{\dfrac{s}{\omega_0}} \tag{23}$$

where:

$$\omega_0 = \frac{V_{out}D'}{L} \tag{24}$$

$$\omega_{z_2} = \frac{R_{load}D'^2}{L} \tag{25}$$

From the above expression, we can see a pole at the origin given by the inductor *L* and a zero featuring a positive root: this is the RHPZ ω_{z_2} we are looking for. Please note that both depend on the duty cycle and are moving in relationship to the input/output conditions.

Applying the boost converter numerical values from our previous example, we have the following positions:

$$f_0 = 1.6\ kHz \tag{26}$$

$$f_{z_2} = 6.6\ kHz \tag{27}$$

In the low frequency domain, for $s << \omega_{z_2}$, the ac output current is dominated by the inductor pole and the phase lags to –90°. The gain drops with a –1 slope until it crosses the 0-dB axis at ω_0. It then continues to further drop until the RHPZ kicks-in. With a LHP zero, the slope would brake from –1 to zero, as it does, but the phase would return to –90° when the frequency further increases. Given the negative sign in Eq. (23), the phase will further lag by –90°, reaching a total of –180° in higher frequencies. We can easily calculate the asymptotic phase limits using Eq. (23):

$$\arg\left[\frac{\hat{i}_{out}(s)}{\hat{d}(s)}\right] = \tan^{-1}\left(-\frac{s}{\omega_{z_2}}\right) - \tan^{-1}(\infty) \tag{28}$$

$$\lim_{s\to 0}\arg\left[\frac{\hat{i}_{out}(s)}{\hat{d}(s)}\right] = \tan^{-1}(0) - \tan^{-1}(\infty) = -90° \tag{29}$$

$$\lim_{s\to \infty}\arg\left[\frac{\hat{i}_{out}(s)}{\hat{d}(s)}\right] = \tan^{-1}(-\infty) - \tan^{-1}(\infty) = -90° - 90° = -180° \tag{30}$$

Our average model from Fig. 4 lends itself very well to plotting Eq. (23). To further check the resulting curves, we have entered this equation into Mathcad® and superimposed both results. As Figure 7 confirms, they are equivalent, showing the phase lag to –180° at high frequency.

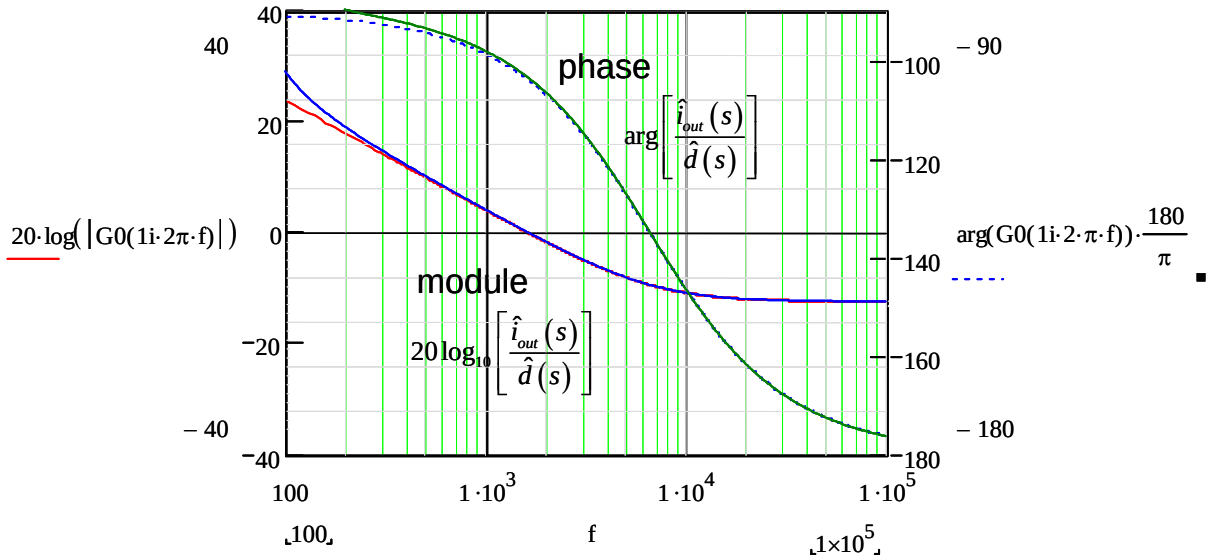

Figure 7: in high frequency, the –90° phase rotation brought by the RHPZ cumulates with that from the origin pole, bringing the total phase rotation to –180°.

The compensation of a system featuring such RHPZ is almost impossible given the phase stress as the crossover frequency approaches the RHPZ position. The only solution is to reduce the bandwidth to 20-30% of the worse case RHPZ position where the total phase stress remains manageable. By reducing the crossover frequency, the resulting duty cycle slew rate stays within acceptable boundaries where the inductor current can always keep up with the demand.

Current mode control

In current mode control, the controller does not directly drive the duty cycle but the inductor peak current. However, as the overall structure of the converter does not change, Eq. (15) remains the same. Since the duty cycle is now a consequence of the inductor peak current set point imposed by the control voltage V_c, let us rework Eq. (18) to extract the duty cycle as a function of the inductor current:

$$\hat{d}(s) = \frac{sL}{V_{out}} \hat{i}_L(s) \tag{31}$$

If now substitute the above equation in Eq. (19), we obtain:

$$\hat{i}_{out}(s) = \hat{i}_L(s)(1-D) - \frac{sL}{V_{out}} I_L \hat{i}_L(s) = \hat{i}_L D' - \hat{i}_L(s) \frac{sL}{D' R_{load}} \tag{32}$$

The ac inductor peak current is imposed by the control voltage across the sense resistor and follows:

$$\hat{i}_L = \frac{\hat{v}_c}{R_{sense}} \tag{33}$$

If we substitute Eq. (33) in Eq. (32) and re-arrange the result, we have:

$$\frac{\hat{i}_{out}(s)}{\hat{v}_c(s)} = \frac{D'}{R_{sense}} - \frac{sL}{D' R_{load} R_{sense}} = G_0 \left(1 - \frac{s}{\omega_{z_2}} \right) \tag{34}$$

where:

$$G_0 = \frac{D'}{R_{sense}} \tag{35}$$

$$\omega_{z_2} = \frac{R_{load} D'^2}{L} \tag{36}$$

Unlike the voltage-mode equation (Eq. (23)), Eq. (34) teaches us the presence of a static gain G_0 independent from the frequency below the RHPZ location. This is the consequence of the current mode technique whose inner current loop removes the inductor pole present in voltage mode control. One immediate comment concerns the RHPZ which is still there in current mode and occupies a same location as with the voltage mode case.

Figure 8 depicts the same boost converter as before but now using a peak current mode controller, also described in Ref. [1]. This new average mode is capable of modelling sub-harmonic instabilities and can toggle between DCM and CCM modes. The control voltage is adjusted to deliver 24 V and it corresponds to a similar duty cycle as before: 58.3%. Applying the boost converter numerical values, the static gain G_0 reaches –7.6 dB. Again, we have entered Eq. (34) in Mathcad and the resulting calculations are plotted in Figure 9 together with the SPICE-simulated waveforms. The agreement is fairly good until the sub-harmonic poles kick-in at half the switching frequency and further degrades the phase response.

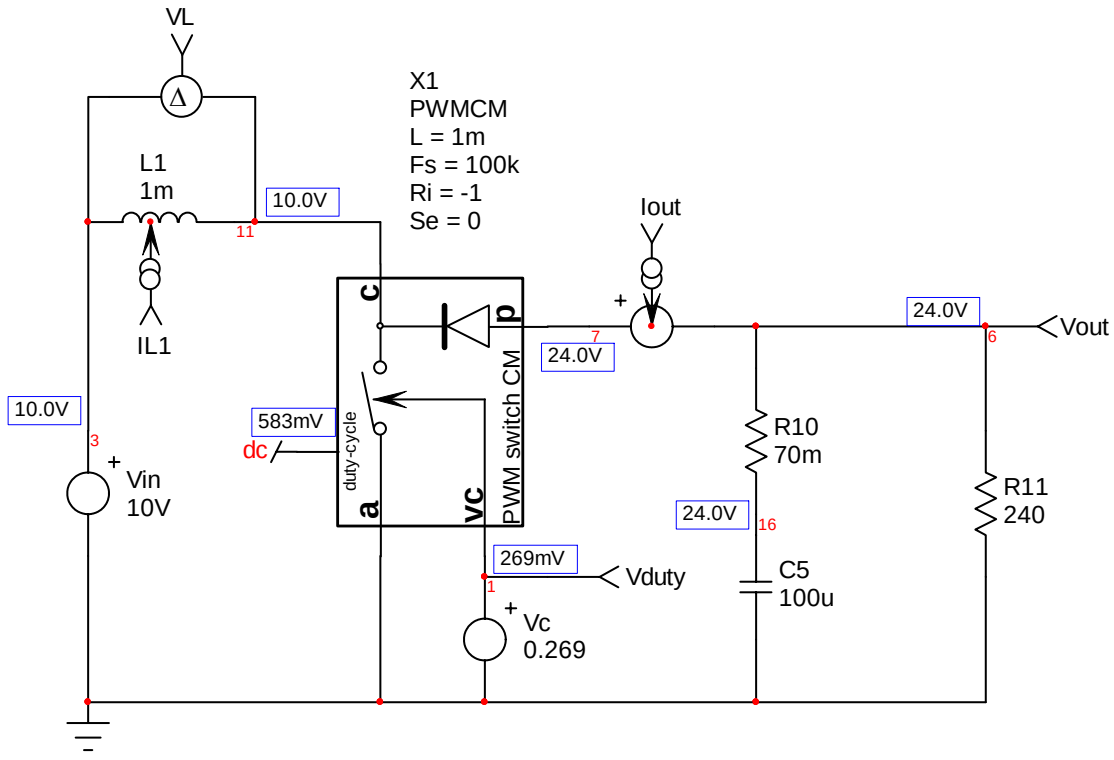

Figure 8: the voltage mode model is now replaced by the current mode sub circuit.

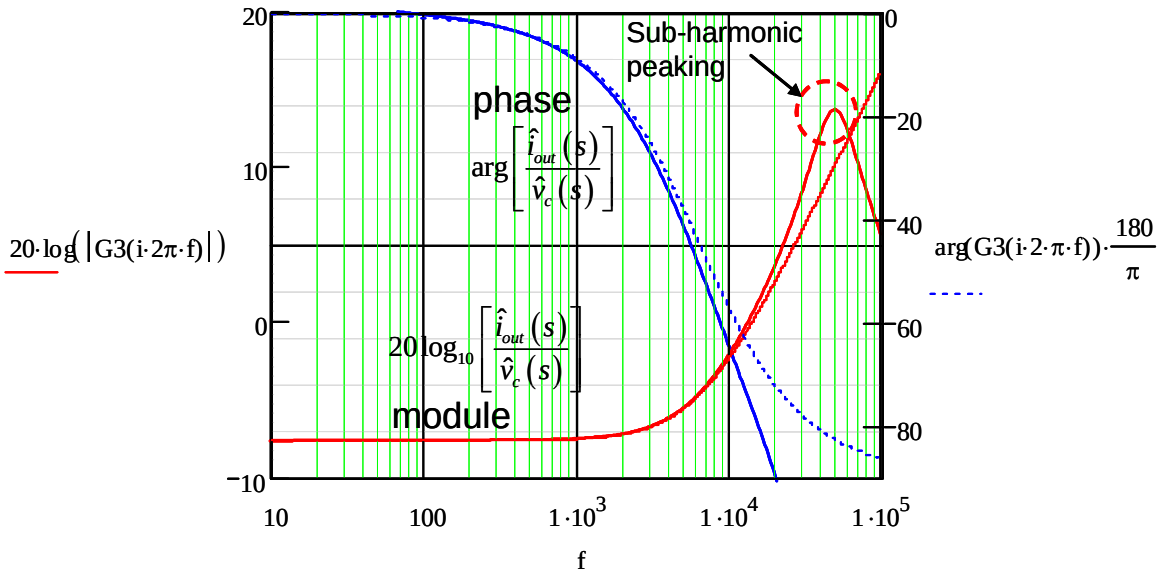

Figure 9: the resulting ac SPICE simulation shows the peaking brought by the sub-harmonic poles. Here, the ramp compensation level is 50% of the inductor current downslope.

Reference [2] offers another interesting way to evaluate the RHPZ position in a boost converter. Using the high-frequency small-signal response of the converter, the author calculates the temporal response of the $\frac{\hat{v}_{out}(s)}{\hat{d}(s)}$ transfer function to a duty cycle transient step. In a separate paragraph, he graphically calculates the

output voltage variation related to the average output current change engendered by a similar abrupt duty cycle change. As both voltages should be equal, the RHPZ is further unveiled in a unique way.

Reference [3] also documents the RHPZ aspects and is worthwhile to consult.

Conclusion

Converters implementing an indirect energy transfer type of conversion suffer from the presence of a right-half-plane zero when operated in CCM. These converters must first store the energy in the inductor during a certain time before dumping it into the output capacitor during the rest of the cycle. If the duty cycle quickly changes in response to a perturbation, the inductor naturally limits the current slew rate and the output voltage drops. A way to limit the vicious effects of the RHPZ is to limit the available loop bandwidth to 20-30% of the worse case RHPZ position. That way, the duty cycle slew rate is limited and remains always slower than the minimum inductor slew rate. The calculations show that the RHPZ exists in CCM fixed-frequency voltage mode and current mode techniques, occupying a similar position. In a next article, we will show how to compensate a converter featuring a RHPZ with the help of SPICE models.

References

1. C. Basso, “Switch Mode Power Supplies: SPICE Simulations and Practical Designs”, McGraw-Hill, 2008
2. V. Vorpérian, “Fast Analytical techniques for ELECTRICAL and ELECTRONIC CIRCUITS”, CAMBRIGE, 0-521-62442-8
3. L. Dixon, “The right-half-plane zero – a simplified explanation”, Unitrode seminars SEM-500